PROBABILITY ON TRANSITIVE GRAPHS

INTRODUCTION

In this course, we will study two random processes on general <u>infinite transitive graphs</u>.

- the simple random walk,
- Bernoulli percolation.

The main idea is to understand the relationship between the geometric properties of the graph and the properties of the random processes. In this introduction we give an overview of the course, introduce the framework (at a non-rigorous level) and give some of the key results that will be proved.

A graph is given by a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges (given by a set of pairs of the form $\{x, y\}$, $x \in V$, $y \in V$). Roughly speaking, a graph is said to be transitive if the graph "looks the same" around every vertex. For example the standard d-dimensional lattice or the d-regular tree are transitive, but the half line $\{0, 1, 2, \dots\}$ with edges between $\{i, i+1\}$ is not transitive.

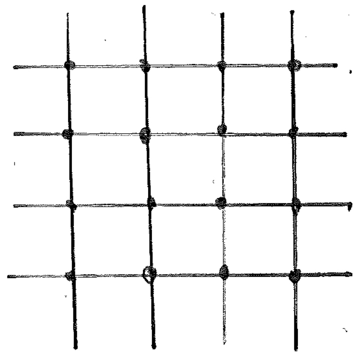
Examples of transitive graphs.

$\mathbb{Z}$

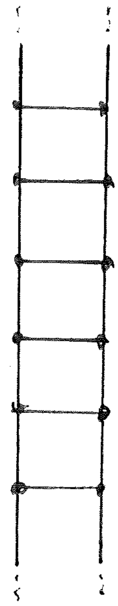
"line"

$\mathbb{Z}^2$

"square lattice"

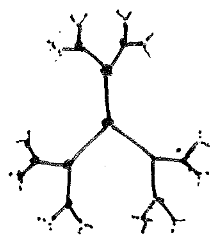


"ladder"



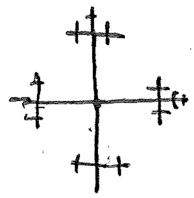
$\mathbb{T}_3$

"3-regular tree"



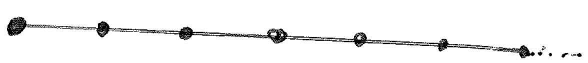
$\mathbb{T}_4$

"4-regular tree"

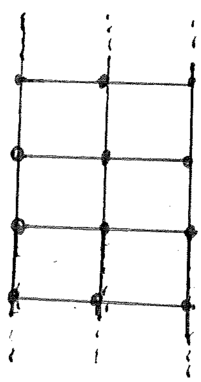


Non transitive graphs.

$\mathbb{N}$



"half line".



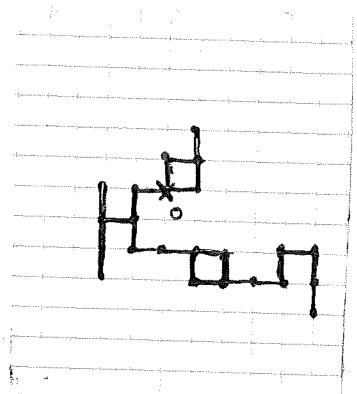
"double ladder graph"

NB: the graphs considered will be infinite with finite degree.

Simple random walk. (SRW)

Consider a graph $G=(V,E)$ with a fixed origin $O\in V$.

A walker starts at the origin O, and at each step, independently of the past, he moves uniformly to one of the neighbours of its position. The set of visited vertices $O=X_0, X_1, X_2, \dots$ is called the simple random walk on G. We will examine several questions regarding the long-time behaviour of the walk.



A walk on $\mathbb{Z}^2$ after 40 steps

Will the random walker visit the origin infinitely often?

On $\mathbb{Z}^d$, a theorem of Polyá answers this question.

Thm [POLYA '21].

- If $d=1,2$, then a.s. X_n visits the origin infinitely many times (the walk is recurrent)
- If $d\geq 3$, then a.s. X_n visits the origin finitely many times (the walk is transient)

On a d-regular tree $(d\geq 3)$ the random walk is always transient. In this course we will study the following closely related question.

What is the return probability after n steps?

Introduce $p_n = \mathbb{P}[X_n = 0]$. The asymptotic behaviour of p_n is very different on $\mathbb{Z}^d$ and on the tree $\mathbb{T}^d$.

Thm:

For the simple random walk on $\mathbb{Z}^d$, $\exists\, c = c(d) > 0$ s.t.

$$\boxed{p_{2n} \underset{n\to\infty}{\sim} \frac{c(d)}{n^{d/2}}}$$ "polynomial decay"

For the simple random walk on $\mathbb{T}^d$, $d \geq 3$,

$$\boxed{p_{2n} \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi n}} \times \left(\frac{4d-4}{d^2}\right)^n}$$ "exponential decay"

For more general transitive graphs, we will be able to characterize the exponential decay of p_n by a simple geometric criterion. Introduce the Cheeger constant

$$\phi = \inf_{S \subset V} \frac{|\vec{\partial S}|}{|S|}$$ "edge boundary"

where the infimum is over all the finite connected subsets of V, and $\partial S = \{ \{x,y\} \in E : x \in S, y \notin S \}$.

Theorem [Kesten]

Consider the SRW on a transitive graph, then

$$\left(\exists c > 0 : \forall n \geq 1 \ p_n \leq e^{-cn} \right) \iff (\phi > 0)$$

At which speed does the random walk escape away from the origin?

Let $|X_n|$ be the graph distance from 0 and X_n. We are interested on the asymptotic behavior of the ratio $\frac{E[|X_n|]}{n}$ as n tends to infinity.

<u>ON $\mathbb{Z}^d$</u> Let $(e_1, \dots, e_d)$ be the canonical basis of $\mathbb{Z}^d$.

The random walk can be written

$$X_n = Z_1 + \cdots + Z_n$$

where $Z_1, \dots, Z_n$ are iid random variables, and Z_i is uniform in $\{\pm e_1, \dots, \pm e_d\}$.

Hence by the law of large number

$$\frac{X_n}{n} \xrightarrow[n\to\infty]{} E[Z_1] = 0$$

And therefore by dominated convergence

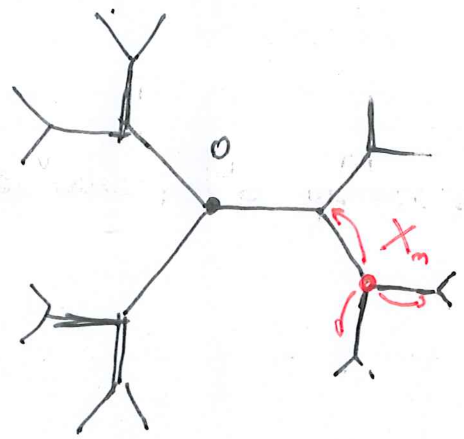
$$\boxed{E\left[\frac{|X_n|}{n}\right] \xrightarrow[n\to\infty]{} 0}$$ "zero speed"

<u>Exercice</u>: Use the central limit theorem to prove

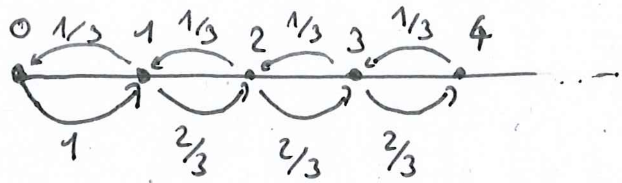
$$\frac{E[|X_n|]}{\sqrt{n}} \xrightarrow[n\to\infty]{} c \quad \text{where } c > 0 \text{ constant.}$$

"diffusive behavior".

On T^3



At each step the walker has probability $\frac{1}{3}$ to get closer to 0, and $\frac{2}{3}$ to move away from the origin (except when the walker is at the origin). Since the jumps are independent $|X_n|$ is a Markov Chain on $\mathbb{N}$ with jump rates given by the following graph:



In particular, we can show that

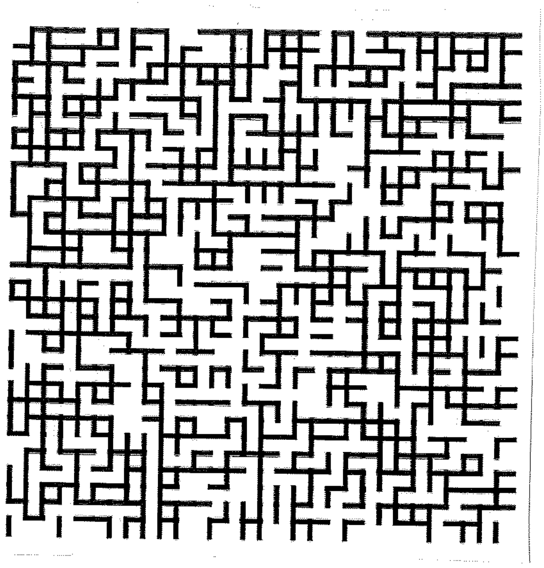
$$\frac{E[|X_n|]}{n} \longrightarrow \frac{1}{3} \qquad \text{"positive speed"}$$

In this course, we will give some "geometric" conditions on the graph G that are equivalent to the positive speed of the random walk.

Percolation.

Bernoulli percolation has been introduced in the 60's by the physicist Hammersley in order to model porous media. Originally studied on d-dimensional lattices, Bernoulli percolation has been considered on more general graphs over the last 20 years. This gave rise to many questions (see, e.g. [Benjamini, Schramm, percolation beyond $\mathbb{Z}^d$, many questions and few answers, '96]) and also many techniques. In particular, we will present the mass-transport principle, which will allow us to prove several results on non-amenable graphs that are not accessible yet on $\mathbb{Z}^d$.

Let $G=(V,E)$ be an infinite transitive graph, and $p\in[0,1]$. Delete each edge with probability $1-p$, independently of the other edges, and consider the set $\omega\subset E$ of remaining edges. We are interested in the connectivity properties of the random subgraph (V,ω) of G.



A percolation configuration on $\mathbb{Z}^2$ $(p = 0{,}55)$

Is there an infinite connected component in ω ?

Let $0 \leftrightarrow \infty$ be the event that 0 belongs to an infinite component of (V, ω).

Write $\mathbb{P}_p$ for the Bernoulli percolation measure when the edges are kept with probability p, and define $p_c = \sup \{ p : \mathbb{P}_p [0 \leftrightarrow \infty] = 0 \}$.

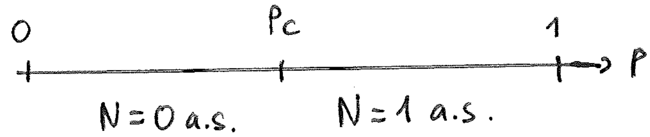
$$0 \quad \mathbb{P}_p[0 \leftrightarrow \infty] = 0 \quad p_c \quad \mathbb{P}_p[0 \leftrightarrow \infty] > 0 \quad 1$$

On $\mathbb{Z}$, due to almost sure existence of cut-points when $p < 1$, we have $p_c = 1$. On $\mathbb{Z}^2$, Kesten [80] proved that $p_c = \frac{1}{2}$ and on $\mathbb{Z}^d$, $d \geq 3$ we know that $0 < p_c < 1$. On the tree $\mathbb{T}^d$, $d \geq 3$ the critical parameter is $p_c = \frac{1}{d-1}$. In this course, we will give some simple conditions ensuring $0 < p_c < 1$ for more general transitive graph. We will also discuss an important conjecture of Benjamini and Schramm ['96] stating roughly that the only transitive graphs with trivial p_c "look like" the line $\mathbb{Z}$.

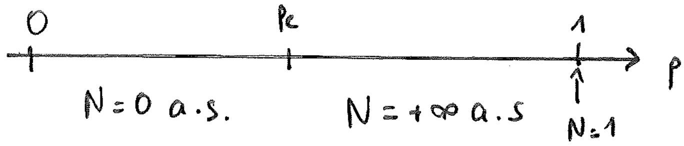
What is the number of infinite components?

For percolation on $\mathbb{Z}^d$, we know that there cannot exist more than one infinite connected component. On general transitive graphs, the picture is richer, and the situation with infinitely many infinite component may be possible. Let us first discuss this phenomenon on some examples. Let $N := N(\omega)$ be the number of infinite connected components in ω, and the following behaviours are known.

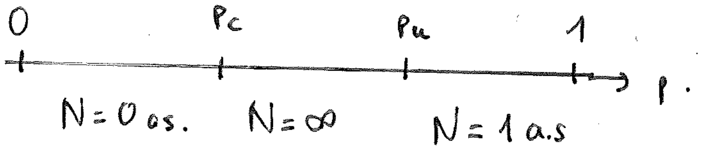
on $\mathbb{Z}^d, d \geq 2$

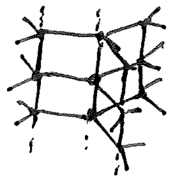


on $\mathbb{T}^d, d \geq 3$



on $\mathbb{T}^d \times \mathbb{Z}$





"there are two phase transitions"

For general graphs we can also define

$$p_u = \inf \{ p : \quad N = 1 \text{ a.s.} \}.$$

(for $\mathbb{Z}^d$ we have $p_c = p_u$ while for $\mathbb{T}^d$ $p_c < p_u = 1$)

We expect that the existence of a phase with infinitely many infinite cluster can be characterized by a simple geometric condition.

Conjecture [Benjamini - Schramm].

$$\boxed{\begin{array}{l}\text{Consider Bernoulli percolation on a connected transitive graph. Then} \\ \left(\exists\, p \in (0,1) \text{ s.t. } \mathbb{P}_p[N = \infty] = 1 \right) \Longleftrightarrow \left(\phi > 0 \right)\end{array}}$$

We will prove some partial results in this direction

First, we will prove $(\phi = 0) \Rightarrow (\mathbb{P}_p[N \in \{0,1\}] = 1)$ which gives one direction of the conjecture and provide families of graphs for which the conjecture holds.

What is the behaviour at p_c?

Conjecture

> Let G be a transitive graph with $p_c < 1$.
> Then for $p = p_c$, we have $\mathbb{P}_p[N = 0] = 1$

This conjecture is proved for $G = \mathbb{Z}^2$ and $G = \mathbb{Z}^d$, $d \geq 11$ but is still open for $G = \mathbb{Z}^3$.

In this course, we will prove the conjecture for Cayley graphs satisfying $\phi > 0$.

CHAPTER 1 : TRANSITIVE GRAPHS

1) DEFINITIONS

Ref : [DIESTEL, Chap. 1] [GODSIL-ROYLE, Chap 1, 3]

Def : A graph is a pair $G = (V, E)$ where
- V is an arbitrary set
- E is a subset of $\{\{x,y\} : x, y \in V\}$.

Notation. For $x, y \in V$ we write $xy := \{x, y\}$.
If $xy \in E$, we say that x and y are neighbours, and we write $x \sim y$.

Notice that the edges are unoriented ($xy = yx$)

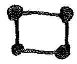
Def : The degree of a vertex $x \in V$ is defined by
$$\deg(x) = \left|\{ y \in V : y \sim x \}\right|.$$
We say that G is locally finite if $\forall x \in V \ \deg(x) < \infty$.

Def : Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $\phi : V \longrightarrow V'$ s.t.
$$(x \sim y \text{ in } G) \iff (\phi(x) \sim \phi(y) \text{ in } G')$$
In this case, we say that ϕ is an isomorphism from G to G'.

Ex :

Diagrammatic representations of two isomorphic graphs

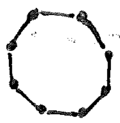
Rk: If ϕ isomorphism from G to G', then $\deg_G(x) = \deg_{G'}(\phi(x))$

$\hookrightarrow$  and  are not isomorphic.

Def: An automorphism of $G = (V, E)$ is an isomorphism from G to itself.

Not: $\mathrm{Aut}(G) = \{\text{automorphisms of } G\}$ group of automorphisms of G
(it is a subgroup of the group of the permutations of V)

Example:

$G = \mathbb{Z}/_{m}\mathbb{Z}$  $\mathrm{Aut}(G) = \{\tau^k, k = 0 \ldots m-1\} \cup \{\tau^k \circ r, k = 0 \ldots m-1\}$
where $\tau(x) = x+1$ and $r(x) = -x$
"diedral group of size $2m$".

Def: A graph $G = (V, E)$ is transitive iff

$$\boxed{\forall x, y \in V \quad \exists \phi \in \mathrm{Aut}(G) \text{ s.t. } \phi(x) = y}$$

Rk: Equivalently, G is transitive if the action of $\mathrm{Aut}(G)$ on V is transitive. (As a subgroup of the permutations of V, $\mathrm{Aut}(G)$ acts naturally on V: for $\phi \in \mathrm{Aut}(G)$, $x \in V$ $\phi \cdot x = \phi(x)$)

Rk: If G is transitive, $\deg(x)$ does not depend on $x \in V$ and is called the degree of G.

2. CAYLEY GRAPHS

Ref: [LYONS-PERES, Section 3.4] [SISTO] [DE LA HARPE, Chapters 2, 4]

Def: Let Γ be a group, let $S \subset \Gamma$
We say that
- S generates Γ if the smallest subgroup of Γ containing S is Γ.
- S is symmetric if $S^{-1} = S$, ie $s \in S \Leftrightarrow s^{-1} \in S$

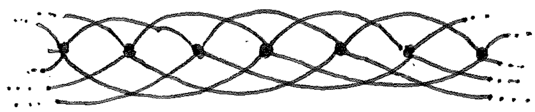
Def: A group Γ is <u>finitely generated</u> if $\exists S \subset \Gamma$ finite that generates Γ.

Def: Let Γ be a finitely generated group. Let $S \subset \Gamma$ be a finite symetric set generating Γ.
The <u>Cayley graph</u> $Cay(\Gamma, S)$ associated to (Γ, S) is the graph with
- vertex set $V = \Gamma$
- edge set $E = \{\{g, gs\} : g \in \Gamma, s \in S\} = \{gg' : g^{-1}g' \in S\}$

<u>Rk</u>:
- $Cay(\Gamma, S)$ is transitive (exercice)
- The definition above corresponds to the right Cayley graph (in the definition of the edges $\{g, g\underset{\uparrow}{s}\}$, the generator "s" on the "right")
Alternatively, one could have defined the left-Cayley graph, which is isomorphic to the right Cayley graph, via $g \to g^{-1}$.

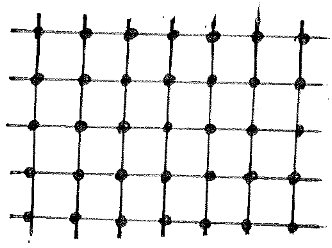
<u>Examples</u>

1) $\Gamma = \mathbb{Z}$ $S = \{-1, +1\}$ $Cay(\Gamma, S) =$ 

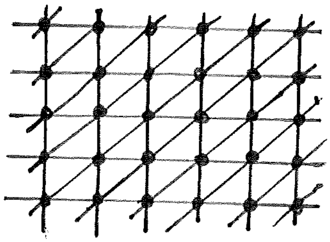
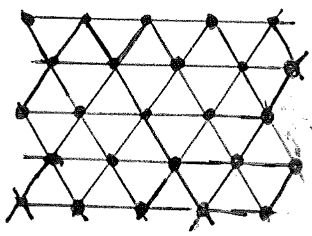
2) $\Gamma = \mathbb{Z}$ $S = \{\pm 2, \pm 3\}$ $Cay(\Gamma, S) =$ 

"changing the generating set gives rise to a different Cayley graph"

3) $\Gamma = \mathbb{Z}^2$ $S = \{\pm(0,1), \pm(1,0)\}$

$Cay(\Gamma, S) =$  "square lattice"

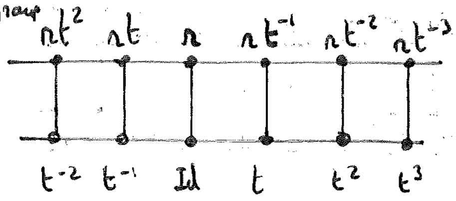
4) $\Gamma = \mathbb{Z}^2$ $S = \{\pm(0,1), \pm(1,0), \pm(1,1)\}$

$Cay(\Gamma, S) =$  $\underset{\substack{\uparrow \\ \text{isomorphic}}}{\cong}$  "triangular lattice"

5) $\Gamma = \mathrm{Aut}(\mathbb{Z})$ $\quad S = \{t^{-1}, t, r\}$ where $t : \begin{array}{l}\mathbb{Z} \to \mathbb{Z} \\ x \mapsto x+1\end{array}$ $\quad r : \begin{array}{l}\mathbb{Z} \longrightarrow \mathbb{Z} \\ x \longmapsto -x\end{array}$

"$\Gamma = D_\infty$ the ∞te diedral group"

$\mathrm{Cay}(\Gamma, S) =$



"ladder graph"

(Notice $rt^{-k} = t^k r$)

<u>Rk</u>: Two different groups may have the same Cayley graph (see e.g. D_∞ and $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$)

<u>Free group over a finite set</u>

<u>Def</u>: Let A be a finite set. Let $S = A \cup \{a^{-1}, a \in A\}$ (a^{-1} is just a formal symbol).

The <u>set of words</u> on S is the set $W(S)$ of finite sequences of elements of S,

$W(S)$ is a monoid : the unit is the empty word e, the product is given by the juxtaposition.

Let $\sim$ be the equivalence relation on $W(S)$ generated by

$w s s^{-1} w' \sim w w'$ and $w s^{-1} s w' \sim w w'$

for every $s \in A$ and every $w, w' \in W(S)$.

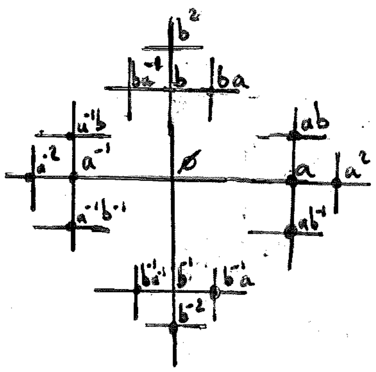
The <u>Free group over A</u> is defined by $\mathbb{F}_A = W(S)/\sim$

<u>Exple</u>: $A = \{a\}$ $\quad S = \{a^{-1}, a\}$ $\quad \mathbb{F}_A = \{a^k, k \in \mathbb{Z}\}$

$\mathrm{Cay}(\mathbb{F}_A, S) =$

$A = \{a, b\}$ $\quad S = \{a^{-1}, a, b^{-1}, b\}$ $\quad$ ($\mathbb{F}_A = \mathbb{F}_2$ free group on two elements)

$\mathrm{Cay}(\mathbb{F}_2, S) =$



"4-regular tree"

Groups defined by generators and relations

Let S be a finite set and $R \subset \mathbb{F}_S$.
The group of presentation $\langle S \mid R \rangle$ is the quotient of $\mathbb{F}_S$ by the normal subgroup generated by R.
$S = \{\text{"generators"}\}$ $\quad R = \{\text{"relators"}\}$
The presentation is said to be finite if S and R are finite.

Examples: $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ $\left(\overset{\text{not.}}{=} \langle a, b \mid ab = ba \rangle \right)$

$\mathbb{Z}/_{n\mathbb{Z}} = \langle a \mid a^n \rangle$ $\left(\overset{\text{not.}}{=} \langle a \mid a^n = 1 \rangle \right)$

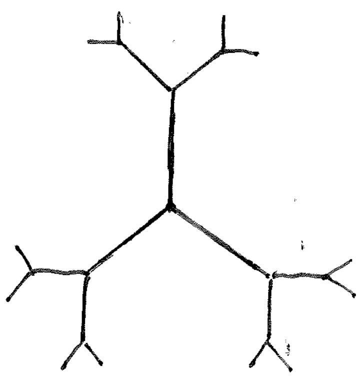
$D_{2m} = \langle t, r \mid t^m, r^2, rtrt \rangle$ $\left(\overset{\text{not.}}{=} \langle t, r \mid t^m = 1, r^2 = 1, rtr^{-1} = t^{-1} \rangle \right)$

$\mathbb{F}_S = \langle S \mid \emptyset \rangle$

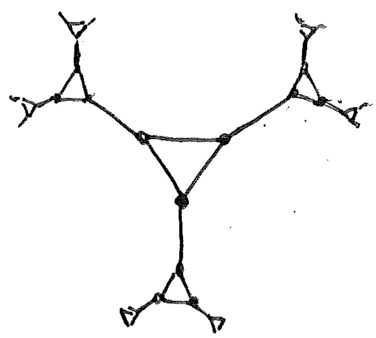
Def. The free product of $\Gamma_1 = \langle S_1 \mid R_1 \rangle$, $\Gamma_2 = \langle S_2 \mid R_2 \rangle, \dots, \Gamma_k = \langle S_k, R_k \rangle$ is the group $\Gamma_1 * \Gamma_2 * \dots * \Gamma_k$ of presentation $\langle S_1 \cup S_2 \cup \dots \cup S_k \mid R_1 \cup R_2 \dots \cup R_k \rangle$

Rk: When a group is defined by a presentation $\langle S \mid R \rangle$ with S finite, it is finitely generated (by the image of S through the quotient). In particular we can define its Cayley graph.

Ex:



$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/_{2\mathbb{Z}} * \mathbb{Z}/_{2\mathbb{Z}}$
$(= \langle a, b, c \mid a^2, b^2, c^2 \rangle)$



$\mathbb{Z}/_{2\mathbb{Z}} * \mathbb{Z}/_{3\mathbb{Z}}$

3) METRIC STRUCTURE

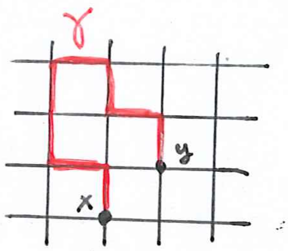
Ref [GODSIL - ROYLE, chap 1]

Let $G = (V, E)$ be a locally finite transitive graph.

Def: A path of length ℓ from a vertex x to a vertex y is a sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_\ell)$ of distinct vertices s.t.

$$\gamma_0 = x, \quad \gamma_\ell = y \quad \text{and } \forall i \quad \gamma_{i-1} \sim \gamma_i .$$

Def. G is said to be connected if for every $x, y \in V$, there exists a path from x to y.



"A path from x to y"

Def: The distance between two vertices x and y is defined by

$$\boxed{d(x,y) = \min_{\gamma : x \to y} \left(\text{length}(\gamma) \right)}$$

where the minimum is over all the paths from x to y.

Rk: The distance d is invariant under the action of $\mathrm{Aut}(G)$:

$$\boxed{\forall \varphi \in \mathrm{Aut}(G) \quad \forall x, y \in V \quad d(\varphi(x), \varphi(y)) = d(x,y).}$$

Pf: Let $\gamma = (\gamma_0, \dots \gamma_\ell)$ be a path from x to y with $\ell = d(x,y)$. Then $\varphi \cdot \gamma = (\varphi \cdot \gamma_0, \dots, \varphi \cdot \gamma_\ell)$ is a path from $\varphi(x)$ to $\varphi(y)$. Hence $d(\varphi(x), \varphi(y)) \leq d(x,y)$. The reverse inequality is obtained by considering φ^{-1}. $\square$

4 GROWTH

Ref: [LYONS - PERES, p.472] [IMRICH - STEIFER]

$G = (V,E)$ infinite, locally-finite, transitive graph. $o \in V$ fixed origin.

Def. For $x \in V$, $n \geq 0$, the ball of radius n around x is defined by

$$\boxed{B_n(x) = \{y \in V : d(x,y) \leq n\}}$$

Not: $B_n = B_n(o)$.

Rk: By transitivity, the graphs induced by $B_n(x)$ and $B_n(y)$ are isomorphic (exercice). In particular $|B_n(x)| = |B_n(y)|$

Prop (Definition of the volume growth exponent)

The following limit exists and is finite:

$$\boxed{v = \lim_{n\to\infty} \frac{1}{n} \log(|B_n|)}$$ "volume growth exponent"

In other words $|B_n| = e^{vn + o(n)}$

Lem (Fekete's subadditivity lemma.)

Let $(u_n)_{n\geq 0}$ be a sequence of numbers in $[-\infty, +\infty)$ satisfying

$$\forall m,n \geq 0 \quad u_{m+n} \leq u_m + u_n \qquad \text{"subadditivity"}$$

Then the limits of $\left(\frac{u_n}{n}\right)$ exists in $[-\infty, +\infty)$

$$\lim_{n\to\infty}\left(\frac{u_n}{n}\right) = \inf_{n\geq 0}\left(\frac{u_n}{n}\right)$$

Proof of the proposition:

We have $B_{m+n} = \bigcup_{x \in B_m} B_x(n)$. Hence $|B_{m+n}| \leq \sum_{x\in B_m} \underbrace{|B_n(x)|}_{=|B_n|}$

Therefore $|B_{m+n}| \leq |B_m| \cdot |B_n|$. By applying Fekete's lemma to $u_n = \log(|B_n|)$ we obtain $\frac{1}{n}\log(|B_n|) \longrightarrow \inf_n \left(\frac{\log(|B_n|)}{n}\right)$

(u_n is finite because G is locally finite.)

<u>Rk</u>: $\forall n \quad |B_n| \geq e^{vn}$.

<u>Rk</u>: We have $\forall n \geq 1 \quad |B_{n+1}| \leq (d-1)\,|B_n|+2$ where d is the degree of G.
Hence $v \leq \log(d-1)$ and the bound is realized for the d-regular tree. (the relation $|B_{n+1}| \leq (d-1)\,|B_n|+2$ can be proved by induction)

<u>Def</u>: We say that G has
- <u>exponential (volume) growth</u> if $v > 0$
- <u>polynomial (volume) growth</u> if $\exists c < \infty$ s.t. $\forall n \quad |B_n| \leq n^c$
- <u>intermediate (volume) growth</u> if $v = 0$ and $\forall c < \infty \quad \sup_n \left(\frac{|B_n|}{n^c}\right) = +\infty$

<u>Examples</u>:
- T_d, $d \geq 3$ has exponential growth
- $\mathbb{Z}^d$, $d \geq 1$ has polynomial growth.
- there exists Cayley graphs of intermediate growth (Grigorchuck groups)

<u>Thm</u>: [Gromov, Trofimov]

If G has polynomial volume growth, then $\exists k \in \mathbb{N}$ $\exists c_1, c_2 > 0$ s.t.

$$\forall n \geq 0 \quad c_1 n^k \leq |B_n| \leq c_2 n^k.$$

- This deep theorem was proved by Gromov in the framework of groups, he showed that a Cayley has polynomial volume growth if and only if the underlying group is virtually nilpotent.
- It was later extended to transitive graphs by Trofimov.

5 ISOPERIMETRIC CONSTANT.

Ref: [LYONS-PERES, Chapter 6] [PATERSON, Introduction]

$G = (V, E)$ infinite locally-finite, transitive graph, $\deg(G) = d$.

Not: If $S \subset V$ finite, we write $\partial S = \{ xy : x \in S, y \in V \setminus S \}$

Def: The isoperimetric constant of G is defined by

$$\boxed{\Phi = \inf_{\substack{S \subset V \\ \text{finite}}} \frac{|\partial S|}{|S|}}$$

We say that G is
- amenable if $\phi = 0$
- non amenable if $\phi > 0$.

Rk: There exist many characterisations of amenability, see [PATERSON].
The definition originates from group theory.
A group Γ is amenable if there exists an invariant mean on Γ, ie a positive linear map $\Lambda \subset \mathrm{Hom}(L^\infty(\Gamma), \mathbb{R})$ of norm 1 and satisfying $\forall x \in \Gamma \ \forall f \in L^\infty(\Gamma) \ \Lambda(x \cdot f) = \Lambda(f)$.
In the context of finitely generated groups the definitions coincide.

Prop: If G is non amenable, then it has exponential growth

Proof: $$|B_{n+1}| = |B_n| + \underbrace{|\{x \in V : d(x, o) = n+1\}|}_{\geq \frac{1}{d} \cdot |\partial B_n|}$$

$$\geq \left(1 + \frac{\phi}{d}\right) |B_n|$$

By induction $|B_n| \geq \left(1 + \frac{\phi}{d}\right)^n$ $\square$

6) THE LAMPLIGHTER GRAPH.

Ref: [LYONS-PERES p.88-89, p.478] [WOESS 2005]

$G=(V,E)$ transitive locally-finite.

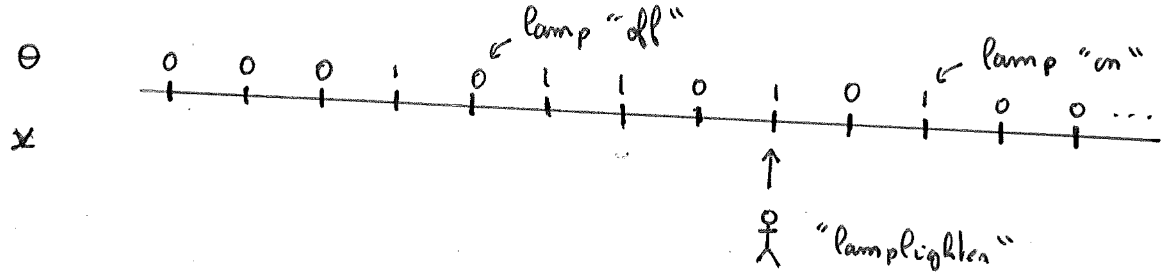
Not. Let Θ be the set of functions $\theta \in (\mathbb{Z}/2\mathbb{Z})^V$ with finite support (ie $\{x : \theta_x \neq 0\}$ is finite)

Def: $LL(G)$ is the graph with vertex set $\Theta \times V$ and edge set defined by

$$(\theta, x) \sim (\theta', x') \iff \begin{cases} x = x' \text{ and } \theta' = \theta + \mathbb{1}_{\{x\}} \\ \text{or} \\ x \sim x' \text{ and } \theta' = \theta \end{cases}$$

Geometric interpretation (for $G = \mathbb{Z}$)

(θ, x) vertex of $LL(G)$



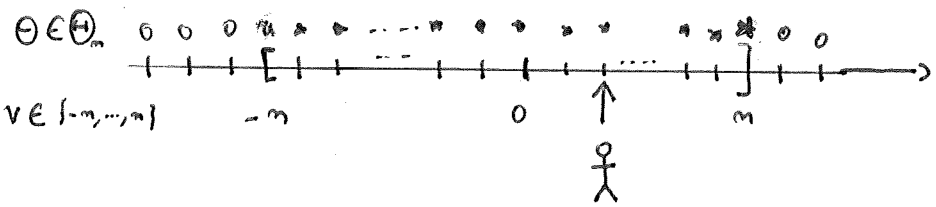
Possible moves: 1) the lamplighter switches on or turns off the lamp at its position.

2) the lamplighter moves to one of its neighbours.

Thm: (i) $LL(\mathbb{Z})$ has exponential volume growth.

(ii) $LL(\mathbb{Z})$ is amenable

<u>Proof</u>: Let $\Theta_n = \{\theta \in \Theta : \theta_x = 0 \ \forall |x| \geq n\}$, and $S_n = \Theta_n \times \{-n, \dots, n\}$.



We have $|S_n| = 2^{2n+1} \times (2n+1)$ and $S_n \subset B_{5n+1}$

Hence $|B_{5n+1}| \geq 2^{2n+1}$

This concludes (i).

Now $\partial S_n = \{\{(\theta,-n),(\theta,-n-1)\}, \theta \in \Theta_n\} \cup \{\{(\theta,n),(\theta,n+1)\}, \theta \in \Theta_n\}$

$$|\partial S_n| = 2 \times 2^{2n+1}$$

And therefore $\dfrac{|\partial S_n|}{|S_n|} = \dfrac{2}{2n+1} \xrightarrow[n\to\infty]{} 0$ which proves (ii) ■

<u>Rk1</u>: The volume growth exponent of $LL(\mathbb{Z})$ can be computed exactly; Using that the set of vertices at distance exactly from 0 "at the right of 0" is the Fibonacci sequence, one can show that $v = \log\left(\frac{1+\sqrt{5}}{2}\right)$, (see exercices)

<u>Rk2</u>: More generally if G is infinite $LL(G)$ has exponential volume growth and (G amenable) $\Leftrightarrow$ ($LL(G)$ amenable)

<u>Rk3</u> If (Γ, S) is a finitely generated group, there exists a group denoted $\Gamma \wr \mathbb{Z}/2\mathbb{Z}$ and a natural generating

"wedge product"

set $\tilde{S}$ s.t. $\mathrm{Cay}(\Gamma \wr \mathbb{Z}/2\mathbb{Z}, \tilde{S}) = LL(\mathrm{Cay}(\Gamma, S))$.

"lamplighter group" see [LYONS PERES p.88]

PART 1

THE SIMPLE RANDOM WALK –

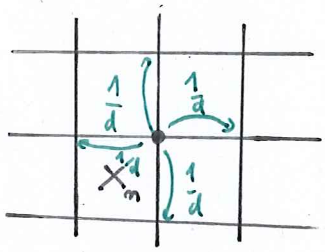
CHAPTER 0

DEFINITIONS AND FIRST PROPERTIES

$G=(V,E)$ transitive, locally-finite, connected, infinite graph.
degree d, fixed origin $o \in V$.

1. DEFINITION

Def: Let $x \in V$. The simple random walk (SRW) on G starting at x is the homogeneous Markov Chain $(X_n)_{n\geq 0}$ with
- state space V
- initial state $X_0 = x$
- transition probabilities $p(x,y) = \begin{cases} \frac{1}{d} & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases}$



"At each step the walker jumps from its position to one of its neighbours, chosen uniformly and independently of the previous steps."

Rk: If $G=(V,E) = Cay(\Gamma, S)$ where S is a finite symmetric set generating the group Γ, then the random walk on G can be defined as follows. Let $Z_1, Z_2, \dots$ be an iid sequence of uniform random variables on S. Then (X_n), defined by

$$X_n = x \cdot Z_1 \cdot Z_2 \cdot \ldots \cdot Z_n,$$

is a simple random walk on G, starting at x.

Convention: For every x, we consider a probability space $(\Omega, \mathbb{P}_x)$ such that, under $\mathbb{P}_x$, $(X_n)_{n\geq 0}$ is a SRW starting at x.

Not. For $m \geq 0$, $x, y \in V$, we write $\boxed{p_m(x,y) = \mathbb{P}_x[X_m = y]}$
and $\boxed{p_m(x) = p_m(0,x)}$

2. ELEMENTARY PROPERTIES

Ref: [ALDOUS-FILL]

2.1 SIMPLE MARKOV PROPERTY

Prop. Let $f : V^{\mathbb{N}} \to \mathbb{R}$ measurable bounded. For every $m \geq 0$, $x, y \in V$

$$\mathbb{E}_x\left[f\left((X_{m+n})_{n\geq 0}\right) \mid X_0, \dots, X_m\right] \underset{\uparrow}{=} \mathbb{E}_y\left[f\left((X_n)_{n\geq 0}\right)\right] \text{ on } \{X_m = y\}$$

$\mathbb{P}_x$-a.s.

"Conditional on $\{X_m = y\}$, $(X_{m+n})_{n \geq 0}$ is a SRW starting at y, independent of $X_0, \dots, X_{m-1}$"

Consequences.

(i) $\forall m, n \geq 0$ $\forall x, y \in V$

$$\boxed{p_{m+n}(x,y) = \sum_{z \in V} p_m(x,z)\, p_n(z,y)} \quad \text{[Chapman - Kolmogorov]}$$

(ii) $\forall n \geq 0$ $\forall x_0, x_1, \dots, x_n \in V$

$$\mathbb{P}_x[X_0 = x_0, \dots, X_n = x_n] = 1_{x = x_0}\, p(x_0, x_1) \dots p(x_{n-1}, x_n)$$

→ "matrix product interpretation"

Rk: A walk from x of length n is a sequence $\gamma = (\gamma_0, \dots, \gamma_n)$ such that $\gamma_0 = x$ and $\forall i$ $\gamma_{i-1} \sim \gamma_i$. (Contrary to a path, the vertices of a walk are not necessarily disjoint).

By (ii)
$$\mathbb{P}_x[(X_0, \dots, X_n) = (x_0, \dots, x_n)] = 1_{x = x_0} \frac{1}{d^n} 1_{x_0 \sim x_1} \dots 1_{x_{n-1} \sim x_n}$$
$$= \frac{1}{d^n} 1_{\{(x_0, \dots, x_n) \text{ walk from } x \text{ of length } n\}}$$

"$(X_0, \dots, X_n)$ is a uniformly chosen walk from x of length n."

2.2 IRREDUCIBILITY

Prop: The SRW is an irreducible Markov Chain:

$$\forall x, y \in V \quad \exists m \geq 0 \ : \ p_m(x,y) > 0$$

Proof: Since $G = (V, E)$ is connected, there exists a path $\gamma = (\gamma_0, \dots, \gamma_m)$ from x to y.

$$p_m(x,y) = \mathbb{P}_x [X_m = y]$$

$$\geq \mathbb{P}_x [X_0 = \gamma_0, \dots, X_m = \gamma_m]$$

$$= \frac{1}{d^m} > 0 \qquad \blacksquare$$

2.3 APERIODICITY.

Recall that the period of (X_n) is defined as

$$\gcd \{ n \geq 0 : p_n(0) > 0 \}$$

Since $\forall k \geq 0 \ \ p_{2k}(0) > 0$, the period of the SRW is 1 (aperiodic case) or 2. The following proposition gives geometric conditions on G characterizing the aperiodic case.

Def: A cycle is a path $\gamma = (\gamma_0, \dots, \gamma_n)$ s.t. $\gamma_0 = \gamma_n$

Prop: The following are equivalent.

(i) the SRW is aperiodic

(ii) there exists an odd cycle in G

(iii) G is not bipartite.

If (i)–(iii) do not hold, then the SRW has period 2.

Proof: (i) $\Rightarrow$ (ii)

If the SRW is aperiodic, there exists n odd such that $\mathbb{P}_0 (X_n = 0) > 0$. Hence $\exists (x_0, \dots, x_n)$ s.t. $x_0 = x_n = 0$ and $\mathbb{P}_0 (X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) > 0$

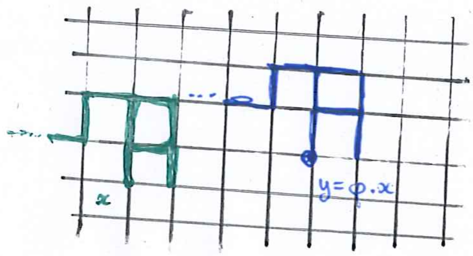
Therefore $(x_0, \dots, x_n)$ is an odd walk satisfying $x_0 = x_n$, and G contains an odd cycle (exercise).

2.4 INVARIANCE

Propn: $\boxed{\forall \varphi \in \mathrm{Aut}(G) \quad p(\varphi\cdot x, \varphi\cdot y) = p(x,y)}$

Consequence: • $\forall n \quad p_n(\varphi\cdot x, \varphi\cdot y) = p_n(x,y)$

• Let $\varphi \in \mathrm{Aut}(G)$ be such that $\varphi\cdot x = y$. If $(X_n)_{n\geq 0}$ is a SRW starting at x, then $(\varphi\cdot X_n)_{n\geq 0}$ is a SRW starting at y.



"in $\mathbb{Z}^2$, the translate of a SRW from x is a SRW from y"

2.5 REVERSIBILITY

Propn: $\boxed{\forall x, y \in V \qquad p(x,y) = p(y,x)}$

"the measure μ on V defined by $\mu_x = 1 \ \forall x \in V$ is reversible for the SRW"

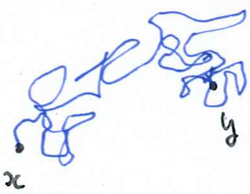
We say that the SRW is reversible (see [Aldous - Fill, Chapter 3])

Consequences / geometric interpretation.

• $\forall n \geq 0 \quad \forall x, y \in V \quad p_n(x,y) = p_n(y,x)$

• $\forall (\overset{x}{\overset{\shortparallel}{x_0}}, x_1, \dots, \overset{y}{\overset{\shortparallel}{x_n}}) \in V^{n+1}$

$$\mathbb{P}_x[(X_0,\dots,X_n) = (x_0,\dots,x_n)] = \mathbb{P}_y[(X_0,\dots,X_n) = (x_n,\dots,x_0)]$$



"the law of a n-step walk from x to y is that of a n-step walk from y to x."

Application:

$$\boxed{\forall m \geq 0 \quad \forall x \in V \quad \text{we have} \quad p_{2m}(0,0) \geq p_{2m}(0,x)}$$

Proof:

$$p_{2m}(0,0) = \sum_{y \in V} p_m(0,y)\, p_m(y,0) \qquad \text{(Chapman - Kolmogorov)}$$

$$= \sum_{y \in V} p_m(0,y)^2 \qquad \text{(Reversibility)}$$

$$= \sqrt{\sum_{y \in V} p_m(0,y)^2}\ \sqrt{\sum_{y \in V} p_m(x,y)^2} \qquad \text{(Invariance + transitivity of } G\text{)}$$

$$\geq \sum_{y \in V} p_m(0,y)\, p_m(x,y) \qquad \text{(Cauchy - Schwarz)}$$

$$= p_{2m}(0,x) \qquad \text{(Chapman - Kolmogorov + Reversibility)} \quad \blacksquare$$

2.6 STRONG MARKOV PROPERTY.

Let $(X_n)_{n \geq 0}$ be a SRW starting at x, $\mathcal{F}_m = \sigma(X_0, \dots, X_m)$, $\mathcal{F}_\infty = \sigma((X_m)_{m \in \mathbb{N}})$.

A stopping time T is a random variable $T \in \mathbb{N}$ s.t. $\forall m$ $\{T = m\} \in \mathcal{F}_m$.

Define $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T = m\} \in \mathcal{F}_m, \text{ for each } m \geq 0\}$.

Prop: Let $f : V^{\mathbb{N}} \to \mathbb{R}$ measurable bounded. For every $x, y \in V$,

$$E_x[f((X_{T+m})_{m \geq 0}) \mid \mathcal{F}_T] = E_y[f((X_m)_{m \geq 0})] \quad \text{on } \{T < \infty, X_T = y\}$$

"Conditioned on $T < \infty$ and $X_T = y$, X_{T+m} is a SRW starting at y, independent of $\mathcal{F}_T$"

2.7. RECURRENCE / TRANSIENCE.

We say that the SRW on G is

- recurrent if $\mathbb{P}_0[X_n = 0 \text{ for infinitely many } n] = 1$
- transient if $\mathbb{P}_0[X_n = 0 \text{ for finitely many } n] = 1$

Let $T_0 = \inf\{n > 0 : X_n = 0\}$ $(\inf \emptyset = +\infty)$

Thm: The following are equivalent.

(i) the SRW is transient

(ii) $\mathbb{P}_0[T_0 = +\infty] > 0$

(iii) $\sum_{n=0}^{\infty} p_n(0) < \infty$

CHAPTER 1 :

RETURN PROBABILITY.

Ref: [LYONS - PERES]

$G = (V, E)$ transitive, loc.-finite, connected, infinite graph degree d (except when $G = \mathbb{Z}^d$), fixed origin $o \in V$.

The goal of this chapter is to prove the following theorem:

Thm [Kesten '58]

> The following are equivalent:
>
> (i) $\exists c > 0$ s.t. $\forall n \geq 0$ $\mathbb{P}_o[X_n = o] \leq e^{-cn}$;
>
> (ii) G is non-amenable.

1) SPECTRAL RADIUS.

1.1) Transition operator.

Motivation: If G finite graph

transition matrix: $(P_{xy})_{x,y \in V} = (p(x,y))_{x,y \in V}$

$\updownarrow$

associated linear operator

$$P : \mathbb{R}^V \longrightarrow \mathbb{R}^V$$
$$(f_x)_{x \in V} \longmapsto \Big(\underbrace{\sum_{y \in V} p(x,y) f_y}_{E_x[f(X_1)]}\Big)_{x \in V}$$

<u>Not</u>: $\boxed{\mathcal{C}_K = \{ f : V \to \mathbb{R} \text{ with finite support} \}}$

For $f, g \in \mathcal{C}_K$ $\quad \langle f, g \rangle = \sum_{x \in V} f(x) g(x)$. $\qquad \|f\|_2 = \sqrt{\langle f, f \rangle}$

<u>Def</u>: The <u>transition operator</u> $P : \mathcal{C}_K \to \mathcal{C}_K$ associated to the SRW on G is defined by

$$\boxed{\forall f \in \mathcal{C}_K \quad \forall x \in V \quad (Pf)(x) = \mathbb{E}_x \left[f(X_1) \right]}$$

<u>Rk</u>: equivalently $(Pf)(x) = \sum_{y \in V} p(x,y) f(y) = \frac{1}{d} \sum_{y \sim x} f(y)$.

<u>Probabilistic interpretation</u>.

If $\pi = (\pi(x))_{x \in V}$ is a finitely supported probability measure on V.
If X_0 is sample according to π then $P\pi$ is the law of X_1 obtained from X_0 by doing one step of the SRW.

In particular for $x, y \in V$

$$\boxed{p_n(x,y) = \langle P^n 1\!\!1_{\{x\}}, 1\!\!1_{\{y\}} \rangle} \quad (*)$$

<u>Not</u>: $\|P\|_2 = \sup_{f \in \mathcal{C}_K \setminus \{0\}} \left\{ \frac{\|Pf\|_2}{\|f\|_2} \right\}$ $\qquad$ " ℓ^2 - operator norm "

<u>Rk</u>: The operator P can be defined on $\ell^2(V)$.

<u>Properties</u>: (i) P is a self-adjoint operator on $\mathcal{C}_K$:

$$\boxed{\forall f, g \in \mathcal{C}_K \quad \langle f, Pg \rangle = \langle Pf, g \rangle}$$

(ii) $\|P\|_2 = \sup_{f \in \mathcal{C}_K \setminus \{0\}} \left\{ \frac{|\langle Pf, f \rangle|}{\langle f, f \rangle} \right\} = \sup_{f \in \mathcal{C}_K \setminus \{0\}} \left\{ \frac{\langle Pf, f \rangle}{\langle f, f \rangle} \right\}$

"Rayleigh Quotient"

Proof: (i) $\langle f, Pg \rangle = \sum_{x \in V} f(x) (Pg)(x)$

$$= \sum_{x,y \in V} p(x,y) f(x) g(y)$$

$$= \sum_{x,y \in V} p(y,x) f(x) g(y) \quad \text{(Reversibility)}$$

$$= \sum_{y \in V} (Pf)(y) \, g(y)$$

$$= \langle Pf, g \rangle$$

(ii) see [Lyons - Peres], ex. 6.7 □

1.2 SPECTRAL RADIUS

Def: The spectral radius of the SRW on G is defined by

$$\boxed{\rho(G) = \limsup_{n \to \infty} \left(p_n(0)^{\frac{1}{n}} \right)}$$

Rk: In particular $p_n(0) \leq \rho(G)^{n + o(n)}$

("exponential decay of the return probability") $\Leftrightarrow$ $(\rho(G) < 1)$

Rk: By irreducibility, we have $\forall x, y \in V$

$$\boxed{\rho(G) = \limsup_{n \to \infty} p_n(x,y)^{\frac{1}{n}}}$$

Rk: The "limsup" is important if the graph G is bipartite

$(p_{2n+1}(0) = 0)$

Exercises: • Prove that $\rho(G) = \lim_{n \to \infty} \left(p_{2n}(0)^{1/2n} \right)$

and if G is not bipartite $\rho(G) = \lim_{n \to \infty} \left(p_n(0)^{1/n} \right)$.

• Prove that $\frac{1}{d} \leq \rho(G) \leq 1$.

Theorem:

$$\boxed{\rho(G) = \|P\|_2 \quad \text{and} \quad \forall n \ \forall x,y \quad p_n(x,y) \leq \|P\|_2^n}$$

We will need the following elementary lemma.

Lemma:

(1) If (u_n) is an increasing sequence of real numbers then $\ell = \lim_{n\to\infty} u_n$ exists and $\ell = \lim_{n\to\infty} \left(\prod_{k\leq n} u_k\right)^{\frac{1}{n}}$. (Cesaro)

(2) Let $(u_n^{(1)}), \dots, (u_n^{(k)})$ be sequences of nonnegative real numbers

$\forall a_1, \dots, a_k > 0 \quad \limsup_{n\to\infty} \left(a_1 u_n^{(1)} + \dots + a_k u_n^{(k)}\right)^{\frac{1}{n}} = \max_k \left(\limsup \left(u_n^{(k)}\right)^{\frac{1}{n}}\right)$

Proof. $p_n(x,y) = \langle P^n \mathbb{1}_x , \mathbb{1}_y \rangle$

$\leq \|P^n \mathbb{1}_x\|_2 \, \|\mathbb{1}_y\|_2$ (By Cauchy-Schwarz).

$\leq \|P\|_2^n$.

This implies $\underline{\rho(G) \leq \|P\|_2}$ and the second part of the theorem.

It remains to prove $\|P\|_2 \leq \rho(G)$.

Let $f \in \mathcal{C}_k \setminus \{0\}$ $f \geq 0$

$\|P^{n+1} f\|_2^2 = \langle P^{n+1} f, P^{n+1} f \rangle$

$= \langle P^n f , P^{n+2} f \rangle$

$\overset{CS}{\leq} \|P^n f\|_2 \, \|P^{n+2} f\|_2$

Hence $\dfrac{\|P^{n+1} f\|_2}{\|P^n f\|_2} \leq \dfrac{\|P^{n+2} f\|_2}{\|P^{n+1} f\|_2}$.

Applying (1) in the lemma to $u_n = \dfrac{\|P^n f\|_2}{\|P^{n-1} f\|_2}$

we have $u_1 \leq \lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\dfrac{\|P^n f\|}{\|f\|}\right)^{1/n} = \lim_{n\to\infty} \|P^n f\|_2^{1/n}$.

Now $\|P^n f\|_2^{\frac{1}{n}} = \langle P^n f, P^n f\rangle^{\frac{1}{2n}}$

$$= \langle P^{2n} f, f\rangle^{1/2n}$$

$$= \left(\sum_{x,y\in V} f(x) f(y)\, p_{2n}(x,y)\right)^{1/2n} \leq \left(\sum_{x,y\in V} |f(x) f(y)|\, p_{2n}(x,y)\right)^{\frac{1}{2n}}$$

By applying (2) in Lemma, we obtain.

$$\lim_{n\to\infty} \|P^n f\|_2^{\frac{1}{n}} \leq \max_{x,y} \left(\limsup_{n\to\infty} \left(p_{2n}(x,y)\right)^{1/2n}\right)$$

$$= \max_{x} \limsup_{n\to\infty} p_{2n}(x)^{1/2n}$$

$$= \limsup_{n\to\infty} p_{2n}(0)^{1/2n} \qquad (\text{because } p_{2n}(x) \leq p_{2n}(0))$$

$$= \rho(G)$$

Therefore $\frac{\|Pf\|_2}{\|f\|_2} \leq \rho(G)$ which proves $\underline{\|P\|_2 \leq \rho(G)}$ $\square$

13 Gradient operator.

Not: We consider the set of oriented edges:

$$\boxed{\vec{E} = \{(x,y) : xy \in E\}.}$$

If $e = (x,y) \in \vec{E}$, we write $\underline{-e = (y,x)}$.

(a non-oriented edge $xy \in E$ is associated to two oriented edges $e = (x,y)$ and $-e = (y,x)$)

Notation: • Let $\mathcal{A}_K$ be the set of functions $\theta : \vec{E} \to \mathbb{R}$ s.t.

- $\forall e \in \vec{E} \quad \theta(-e) = -\theta(e)$, [antisymmetry]
- $\{e \in \vec{E} : \theta(e) \neq 0\}$ is finite. [finite support]

• $\forall \theta, \psi \in \mathcal{A}_K \quad \langle \theta, \psi \rangle_{\vec{E}} = \frac{1}{2} \sum_{e \in \vec{E}} \theta(e)\psi(e)$.

• $\forall \theta \in \mathcal{A}_K \quad \|\theta\|_2 = \sqrt{\langle \theta, \theta \rangle_{\vec{E}}}$.

Def: [Gradient operator]

We define the operator $\nabla : \mathcal{C}_K \longrightarrow \mathcal{A}_K$ by

$$\forall f \in \mathcal{C}_K \quad \forall (x,y) \in \vec{E} \quad \boxed{(\nabla f)((x,y)) = f(y) - f(x)}.$$

Lemma [relation between P and ∇]

For every $f, g \in \mathcal{C}_K$ we have

$$\langle (Id - P) f, g \rangle = \frac{1}{d} \langle \nabla f, \nabla g \rangle_{\vec{E}}.$$

Rk: $Id - P \overset{\text{def}}{=} \Delta$ "discrete Laplacian"

The equation above can be seen as a discrete integration by parts.

Proof: For $x \in V$
$$[(Id - P) f](x) = f(x) - \sum_{y \in V} p(x,y) f(y)$$
$$= \sum_{y \in V} p(x,y) [f(x) - f(y)]$$

Hence
$$\langle (Id - P) f, g \rangle = \sum_{x,y \in V} p(x,y)\,[f(x) - f(y)]\, g(x)$$

$$= \frac{1}{2} \sum_{x,y \in V} p(x,y)\,[f(x) - f(y)]\, g(x)$$

$$+ \frac{1}{2} \sum_{x,y \in V} \underbrace{p(y,x)}_{= p(x,y)}\,[f(y) - f(x)]\, g(y)$$

$$= \frac{1}{2} \sum_{x,y \in V} \underbrace{p(x,y)}_{= \frac{1}{d} 1\!\!1_{(x,y) \in \vec{E}}}\,[f(x) - f(y)]\,[g(x) - g(y)]$$

$$= \frac{1}{d} \langle \nabla f, \nabla g \rangle_{\vec{E}} .$$ ∎

Proposition: [Variational formula for $\rho(G)$]

We have
$$\boxed{\rho(G) = 1 - \frac{1}{d} \left(\operatorname{Inf}_{f \in \ell_k \setminus \{0\}} \frac{\|\nabla f\|_2}{\|f\|_2} \right)^2 .}$$

Proof:

Using Rayleigh quotient we find

$$\rho(G) = \|P\|_2 = \operatorname{Sup}_{f \in \ell_k \setminus \{0\}} \left\{ \frac{\langle Pf, f \rangle}{\langle f, f \rangle} \right\}$$

$$= 1 - \operatorname{Inf}_{f \in \ell_k \setminus \{0\}} \left\{ \frac{\langle (Id - P) f, f \rangle}{\langle f, f \rangle} \right\}$$

$$\overset{\text{Lemma}}{=} 1 - \operatorname{Inf}_{f \in \ell_k \setminus \{0\}} \left\{ \frac{1}{d} \times \frac{\langle \nabla f, \nabla f \rangle_{\vec{E}}}{\langle f, f \rangle} \right\}$$ ∎

2 ISOPERIMETRIC CONSTANT REVISITED

Not. For $\theta \in \mathcal{A}_K$ write $\boxed{\|\theta\|_1 = \frac{1}{2}\sum_{e\in\vec{E}} \theta(e)}$,

and for $f \in \mathcal{C}_K$ write $\boxed{\|f\|_1 = \sum_{x\in V} f(x).}$

Prop: $\boxed{\Phi = \inf\limits_{f\in\mathcal{C}_K\setminus\{0\}} \dfrac{\|\nabla f\|_1}{\|f\|_1}}$

Proof: $\boxed{\geq}$ Let $S\subset V$ finite non empty, then $1\!\!1_S \in \mathcal{C}_K\setminus\{0\}$ and

- $\|\nabla 1\!\!1_S\|_1 = |\partial S|$
- $\|1\!\!1_S\|_1 = |S|$

Hence $\Phi = \inf\limits_{\substack{S\subset V\\ \text{finite}}} \dfrac{\|\nabla 1\!\!1_S\|_1}{\|1\!\!1_S\|_1} \geq \inf\limits_{f\in\mathcal{C}_K\setminus\{0\}} \dfrac{\|\nabla f\|_1}{\|f\|_1}$.

$\boxed{\leq}$ Let $f\in\mathcal{C}_K\setminus\{0\}$.

For $t>0$, consider $S_t = \{x\in V : |f(x)|>t\}$

S_t is finite (because $f\in\mathcal{C}_K$) and by definition

$$\phi\cdot|S_t| \leq |\partial S_t|,$$

which can be rewritten as

$$\phi\cdot\sum_{x\in V} 1\!\!1_{|f(x)|>t} \leq \sum_{(x,y)\in\vec{E}} 1\!\!1_{f(x)>t\geq f(y)}$$

Integrating from $t=0$ to $t=+\infty$ and using Fubini theorem, we obtain

$$\phi \cdot \underbrace{\sum_{x\in V} \int_0^\infty \mathbb{1}_{|f(x)|>t}\, dt}_{=\|f\|_1} \leq \underbrace{\sum_{(x,y)\in\vec{E}} \int_0^\infty \mathbb{1}_{f(x)>t\geq f(y)}\, dt}$$

$$= \sum_{(x,y)\in\vec{E}} \big(|f(x)| - |f(y)|\big)\, \mathbb{1}_{|f(x)|>|f(y)|}$$

$$= \frac{1}{2} \sum_{(x,y)\in\vec{E}} \big|\,|f(x)| - |f(y)|\,\big|$$

$$\leq \|\nabla f\|_1 .$$

Therefore, $\phi \leq \inf_{f\in \ell_k \setminus\{0\}} \dfrac{\|\nabla f\|_1}{\|f\|_1}$. ■

3 PROOF OF KESTEN'S THEOREM.

We prove the following stronger theorem.

Thm:
$$\boxed{1 - \left(\frac{\phi}{d}\right) \overset{(1)}{\leq} \rho(G) \overset{(2)}{\leq} \sqrt{1 - \left(\frac{\phi}{d}\right)^2}}$$

Proof: (1) $\rho(G) = 1 - \dfrac{1}{d} \inf_{f\in \ell_k\setminus\{0\}} \left\{ \dfrac{\langle \nabla f, \nabla f\rangle_{\vec{E}}}{\langle f, f\rangle} \right\}$

$$\geq 1 - \frac{1}{d} \inf_{\substack{S\subset V \\ \text{finite}}} \left\{ \frac{\langle \nabla \mathbb{1}_S, \nabla \mathbb{1}_S\rangle_{\vec{E}}}{\langle \mathbb{1}_S, \mathbb{1}_S\rangle} \right\} = 1 - \frac{\phi}{d} .$$

(2) Let $f \in \mathcal{C}_k \setminus \{0\}$.

$$\|f\|_2^2 \Phi = \|f^2\|_1 \Phi \leq \|\nabla f^2\|_1$$

$$= \frac{1}{2} \sum_{(x,y)\in\vec{E}} |f(x)-f(y)|\,|f(x)+f(y)|$$

$$\overset{CS}{\leq} \|\nabla f\|_2 \times \sqrt{\frac{1}{2}\sum_{(x,y)\in\vec{E}} f(x)^2+f(y)^2+2f(x)f(y)}$$

$$= \|\nabla f\|_2 \times \sqrt{d\sum_{x\in V} f(x)^2 + \underbrace{\sum_{(x,y)\in\vec{E}} f(x)f(y)}}$$

$$= d \times \sum_{x,y\in V} p(x,y)\, f(x)f(y)$$

$$= d \times \langle Pf, f\rangle$$

$$= \|\nabla f\|_2 \times \sqrt{2d\,\|f\|_2^2 - \|\nabla f\|_2^2}$$

Dividing the expression above by $\|f\|_2^2$ and setting $\alpha = \frac{\|\nabla f\|_2}{\|f\|_2}$,

we obtain $\Phi \leq \alpha \times \sqrt{2d-\alpha^2}$,

i.e. $d^2 - \Phi^2 \geq (d-\alpha^2)^2$.

Dividing by d^2, we finally get $1-\left(\frac{\Phi}{d}\right)^2 \geq \left(1-\frac{1}{d}\frac{\|\nabla f\|_2}{\|f\|_2}\right)^2$.

Taking the sup over $f \in \mathcal{C}_k \setminus \{0\}$ concludes that

$$1-\left(\frac{\Phi}{d}\right)^2 \geq \rho'(G)^2 \qquad \blacksquare$$

CHAPTER 2 :

THE VAROPOULOS - CARNE BOUND.

Ref: [LYONS-PERES] [Blog, T. TAO]

$G = (V, E)$ transitive, locally-finite, connected, infinite graph

degree d, fixed origin $0 \in V$.

$(X_n)_{n \geq 0}$ SRW on G, P transition operator, $p_n(x,y) = \mathbb{P}_x[X_n = y]$.

Motivation: In the previous Chapters, we have seen

$$p_n(x,y) \leq p_n(0,0) \leq \|P\|_2^n$$

We expect $p_n(x,y)$ to be be "substancially" smaller than $\|P\|_2^n$ when x and y are far from each other. (In particular, we trivially have $p_n(x,y) = 0$ if $d(x,y) > n$)

In this chapter, we prove the following inequality, which improves quantitatively the bound $p_n(x,y) \leq \|P\|_2^n$ when $d(x,y) \gg \sqrt{2n}$. It is a generalization and improvement by Carne (1985) of a result of Varopoulos (1985). The statement below is a refinement of Lyons and Peres (see Thm 13.4 in [LYONS-PERES]).

Thm: For all $x, y \in V$, for all $n \geq 0$, we have:

$$\boxed{p_n(x,y) \leq 2 \|P\|_2^n \exp\left(-\frac{d(x,y)^2}{2n}\right)}$$

[Varopoulos - Carne Bound.]

We will see several applications in the next chapters:

- When G has polynomial growth it implies that $P_0[d(0,X_n) \leq 4\sqrt{n \log n}] \to 1$
- When G has sub-exponential growth, it implies that X_n has speed $\ell = 0$.

1. The case $G = \mathbb{Z}$.

Thm: Let $(X_n)_{n\geq 0}$ be a SRW on $\mathbb{Z}$, starting at 0. Then

$$\boxed{\mathbb{P}_0\,[\,|X_n| \geq d\,] \leq 2\, e^{-d^2/2n}.}$$

Proof: Let $Z_1, Z_2, \dots, Z_n$ be iid random variables s.t. $\mathbb{P}[Z_1 = 1] = \mathbb{P}[Z_1 = -1] = \frac{1}{2}$.

Let $s = \frac{d}{n}$. By symmetry

$$\mathbb{P}_0\,[\,|X_n| \geq d\,] = 2\mathbb{P}\,[\,sZ_1 + \dots + sZ_n \geq s^2 n\,]$$

$$= 2\,\mathbb{P}\left[e^{sZ_1 + \dots + sZ_n} \geq e^{s^2 n}\right]$$

$$\leq 2\left(\frac{E[e^{sZ_1}]}{e^{s^2}}\right)^n \qquad [\text{By Markov inequality + independence}]$$

Using that $E[e^{sZ_1}] = \cosh(s) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} s^{2k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k \times k!} s^{2k} = e^{s^2/2}$,

we finally get $\mathbb{P}_0\,[\,|X_n| \geq d\,] \leq 2\, e^{-ns^2/2}$ ∎

2. Chebyshev polynomials.

Definition [Chebyshev Polynomials]

We consider the sequence of polynomials $(T_n)_{n\in\mathbb{Z}}$ defined by

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+2}(x) = 2x\,T_{n+1}(x) - T_{n-1}(x) \end{cases} \qquad \text{and } T_{-n} = T_n$$

Property: For every $\theta \in \mathbb{R}$ $\boxed{\cos(n\theta) = T_n(\cos\theta)}$

$\deg(T_k) = |k|$ and

Proof: By induction (use $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\,\cos n\theta$) ∎

Properties:

$$\boxed{\begin{array}{ll} \text{(i)} & \forall m \geq 0 \ \forall u \in [-1,1] \quad |T_m(u)| \leq 1 \\ \text{(ii)} & x^m = \sum_{k=-m}^{m} q_m(k)\, T_k(x). \qquad \text{"polynomial identity"} \end{array}}$$

Proof: (i) choose θ s.t. $\cos\theta = u$ then $|T_m(u)| = |T_m(\cos\theta)| = |\cos(m\theta)| \leq 1$.

(ii) It suffices to prove the equality for $x = \cos\theta$, $\theta \in \mathbb{R}$.

$$x^m = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^m = \sum_{l=0}^{m} \underbrace{\frac{1}{2^m}\binom{m}{l}}_{= q_m(2l-m)} e^{i\theta(2l-m)}$$

$$= \sum_{|k| \leq m} q_m(k)\, e^{i\theta k}$$

$$= \sum q_m(k)\, \frac{e^{i\theta k} + e^{-i\theta k}}{2} \qquad [\text{sym. } q_m(-k) = q_m(k)]$$

$$= \sum_{|k|\leq m} q_m(k)\, T_k(x) \qquad \blacksquare$$

3. PROOF OF THE THEOREM.

Lemma: Let $S : \ell_k \to \ell_k$ self-adjoint bounded ($\|S\|_2 < \infty$), let $Q \in \mathbb{R}[x]$.

Then $$\boxed{\|Q(S)\|_2 \leq \max_{u \in [-\|S\|_2, \|S\|_2]} |Q(u)|.}$$

Proof: first proof: Use the spectral theorem for bounded self-adjoint operators (see [Rudin, Functional analysis, p. 308/303]) we have

$$\langle Q(S)f, Q(S)f \rangle = \int_{-\|S\|_2}^{\|S\|_2} Q(u)^2 \, dE_{f,f}(u) \text{ where } dE_{f,f}(u)$$

is the spectral measure of S attached to f.

Second proof (with a reduction to finite dimension)

Consider the finite-dimensional space $\mathcal{C}_m = \{ f \in \mathcal{C}_K : \mathrm{Supp}(f) \subset B_m \}$ equiped with the inner product $\langle f, g \rangle_{\mathcal{C}_m} = \langle f, g \rangle$

Define $S_m : \mathcal{C}_m \to \mathcal{C}_m$ by setting

$$\forall f \in \mathcal{C}_m \qquad (S_m f)(x) = (Sf(x)) \mathbb{1}_{x \in B_m}$$

S_m is a self-adjoint linear operator on the (finite dimensional) Euclidean space $(\mathcal{C}_m, \langle \cdot, \cdot \rangle_{\mathcal{C}_m})$. Indeed

$$\forall f, g \in \mathcal{C}_m \quad \langle S_m f, g \rangle_{\mathcal{C}_m} \underset{g \in \mathcal{C}_m}{=} \langle Sf, g \rangle \underset{S \text{ auto-adjoint}}{=} \langle f, Sg \rangle \underset{f \in \mathcal{C}_m}{=} \langle f, S_m g \rangle_{\mathcal{C}_m}.$$

Therefore there exists an orthonormal basis $(\varphi_1, \dots, \varphi_L)$ of $\mathcal{C}_m$ made of eigenvectors of S_m. Writing λ_ℓ for the eigenvalue associated to φ_ℓ, we obtain for every $f \in \mathcal{C}_K$ and m large,

$$Q(S_m) f = Q(S_m) \Big(\sum_{\ell \leq L} \langle f, \varphi_\ell \rangle \varphi_\ell \Big)$$

$$= \sum_{\ell \leq L} \langle f, \varphi_\ell \rangle \, Q(\lambda_\ell) \varphi_\ell$$

Hence $\| Q(S_m) f \|_2^2 = \sum_\ell \langle f, \varphi_\ell \rangle^2 Q(\lambda_\ell)^2 \leq \Big(\max_{|\lambda| \leq \|S_m\|_2} |Q(\lambda)| \Big)^2 \|f\|_2^2,$

Which implies $\| Q(S_m) \|_2 \leq \max_{|\lambda| \leq \|S_m\|_2} |Q(\lambda)|.$

and the proof follows from the fact that $\|Q(S_m) f\|_2 \xrightarrow[m \to \infty]{} \|Q(S) f\|_2$ and $\|S_m\|_2 \leq \|S\|_2$.

<u>Proof of Varopoulos - Carne bound:</u>

By Property (ii) of the Chebyshev's polynomials, we have

for every $m \geq 1$ $\quad \left(\frac{P}{\|P\|}\right)^m = \sum_{|k| \leq m} q_m(k)\, T_k\left(\frac{P}{\|P\|}\right)$.

key observation: If $d(x,y) > |\ell|$ $\quad \langle P^\ell 1\!\!1_x, 1\!\!1_y \rangle = 0$

and therefore $\quad \langle T_k\left(\frac{P}{\|P\|_2}\right) 1\!\!1_x, 1\!\!1_y \rangle = 0 \quad \forall\, |k| < d(x,y)$

For every $x, y \in V$, $m \geq 1$

$$
\begin{aligned}
p_m(x,y) &= \langle P^m 1\!\!1_x, 1\!\!1_y \rangle \\
&= \|P\|_2^m \left\langle \left(\frac{P}{\|P\|_2}\right)^m 1\!\!1_x, 1\!\!1_y \right\rangle \\
&= \|P\|_2^m \sum_{|k| \leq m} q_m(k) \underbrace{\left\langle T_k\left(\frac{P}{\|P\|_2}\right) 1\!\!1_x, 1\!\!1_y \right\rangle}_{= 0 \text{ if } |k| < d(x,y)} \\
&\leq \|P\|_2^m \sum_{d(x,y) \leq |k| \leq m} q_m(k) \underbrace{\left\| T_k\left(\frac{P}{\|P\|_2}\right) \right\|_2}_{\underset{\text{lem.}}{\leq} \max_{u \in [-1,1]} |T_k(u)| \underset{(i)}{\leq} 1} \\
&\leq 2 \|P\|_2^m \exp\left(-\frac{d(x,y)^2}{m}\right).
\end{aligned}
$$

■

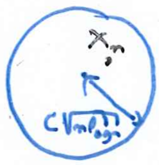
4 ONE APPLICATION.

The Varopoulos-Carne bound provides useful upper bounds on the speed at which the random walk moves away from o. Write $|X_n| = d(o, X_n)$. When the graph G has sub-exponential volume growth, we obtain substantial improvements to the trivial bound $|X_n| \leq n$. In particular we have the following bounds when the growth is polynomial or stretch exponential:

Corollary (to Varopoulos-Carne bound)

(i) If G has polynomial volume growth $(\exists A \ |B_n| \leq A n^D)$ then $\exists C < \infty$ s.t.

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log n}} \leq C \qquad \mathbb{P}_o\text{-a.s.}$$



(ii) If $|B_n| \leq A e^{C n^\alpha}$ for some $A, C > 0$ and $0 < \alpha \leq 1$, then

$$\limsup \frac{|X_n|}{n^{1/(2-\alpha)}} \leq (2C)^{1/(2-\alpha)} \qquad \mathbb{P}_o\text{-a.s.}$$

Proof: (i) Assume $|B_n| \leq A n^D$. Let $N = C\sqrt{n \log n}$

$$\mathbb{P}[|X_n| \geq N] \leq \sum_{k \geq N} \mathbb{P}_o[|X_n| = k]$$

$$\leq \sum_{k \geq N} |B_k| \cdot 2 \exp\left(-\frac{k^2}{2n}\right)$$

$$\leq 2A \underbrace{\sum_{k \geq N} k^D \exp\left(-\frac{k^2}{2n}\right)}_{\leq C_1 N^{A+1} \exp\left(-\frac{N^2}{2n}\right)}$$

Hence $P\left[|X_n| \geq C\sqrt{n \log n} \right] \leq 2AC_1 \times (n \log n)^{\frac{D+1}{2}} n^{-\frac{C^2}{2}}$

If $C > \sqrt{D+3}$ $\quad \sum_n \mathbb{P}_o\left[\frac{|X_n|}{\sqrt{n \log n}} \geq C \right] < \infty$

And Borel-Cantelli lemma concludes that

$$\limsup_{n \to \infty} \frac{|X_n|}{\sqrt{n \log n}} \leq C \quad \text{a.s.}$$

(ii) exercice.

Remarks:

- In (i), it is possible to choose $C = \sqrt{D+1}$ (exercice)
- The bound (i) is not sharp for $G = \mathbb{Z}^d$. Indeed by the law of the iterated logarithm

$$\limsup_{n \to \infty} \frac{|X_n|}{\sqrt{n \log \log n}} < \infty .$$

CHAPTER 3 :

RECURRENCE / TRANSIENCE

Ref: [LYONS - PERES, Section 6.7] [WOESS, Theorem 3.24]

$G = (V, E)$ transitive, loc. finite, connected, infinite graph.
degree d, fixed origin $o \in V$.

In this chapter we will obtain general bounds on $p_n(o,o)$ depending on the "isoperimetric profile" of the graph. As a consequence we obtain the following theorem, which determines for which graphs the SRW is recurrent / transient.

Theorem:

(i) If the growth of G is at most quadratic (ie $\exists C: |B_n| \leq C n^2$), then the SRW is recurrent.

(ii) If the growth of G is at least cubic (ie $\exists c > 0: |B_n| \geq c n^3$), then the SRW is transient.

Remark: By Gromov - Trofimov theorem, any transitive satisfies either (i) or (ii). Therefore the theorem above provides us with a geometric characterization of recurrence / transience.

1. PROOF OF (i)

There are several ways to prove this statement, see e.g. [LYONS - PERES, exercise 2.86] for a proof using Nash-Williams criterion. Here we will prove it using Varopoulos - Carne's bound.

Proof of (ii).

Let C_0 s.t. $\forall n$ $|B_n| \leq C_0 n^2$

By the corollary of Varopoulos - Carne bound, we have for n large enough.

$$\frac{1}{2} \leq \mathbb{P}_0[X_{2n} \in B_N] \quad \text{where } N = \sqrt{C n \log n} \quad (C>0 \text{ constant})$$

$$\leq \sum_{x \in B_N} P_0[X_{2n} = x]$$

$$\leq |B_N| \, p_{2n}(0,0)$$

$$\leq C_0 \cdot C n \log n \, p_{2n}(0,0)$$

Hence $p_{2n}(0,0) \geq \frac{C \cdot C_0}{n \log n}$, which implies that $\sum_{k \geq 0} p_k(0,0) = +\infty$.

Therefore the SRW is recurrent. ∎

2 EXPANSION PROFILE.

Def: The expansion profile of G is the function φ defined by

$$\forall u \geq 1 \quad \varphi(u) = \inf_{1 \leq |S| \leq u} \left(\frac{|\partial S|}{|S|} \right)$$

Rk: φ is non increasing and $\lim_{u \to \infty} \varphi(u) = \Phi$ isoperimetric constant.

In particular (G is amenable) $\Leftrightarrow$ ($\lim \varphi = 0$).

Thm: Define for every $m \geq 1$ $R(m) = \min\{n : |B_n| \geq m\}$.
There exists a constant $c > 0$ s.t.

$$\forall u \geq 1 \quad \varphi(u) \geq \frac{c}{R(2u)}$$

Examples : • If $G = \mathbb{Z}^d$, then we get $R(n) \asymp n^{1/d}$

and we obtain for every $S \subset \mathbb{Z}^d$ finite

$$\frac{|\partial S|}{|S|} \geq \frac{c}{|S|^{1/d}} \quad \text{ie } |\partial S| \geq c\,|S|^{d-1/d}.$$

(The bound is sharp, up to constant)

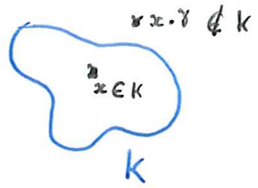
• If G has exponential volume growth then $R(n) \asymp \log n$

$$\forall S \subset V \quad \frac{|\partial S|}{|S|} \geq \frac{c}{\log(S)}$$

(The bound is not sharp if the graph is non amenable).

Proof: Case $G = (V, E) = \mathrm{Cay}(\Gamma, S)$
The general case will be treated later using the mass-transport principle.
Let $K \subset V$ finite non empty. We will prove that

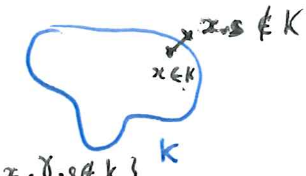
$$\frac{|\partial K|}{|K|} \geq \frac{1}{2R(2|K|)}$$



For $\gamma \in \Gamma$ let $A_\gamma = \{x \in K : x.\gamma \notin K\}$

First observe that for $s \in S$ (generating set)

$$|A_s| = |\{x \in K : x.s \notin K\}| \leq |\partial S|.$$



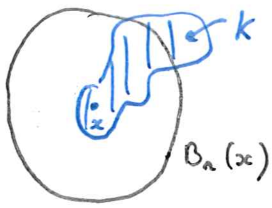
Since $A_{\gamma.s} \subset \underbrace{\{x \in K : x.\gamma \notin K\}}_{A_\gamma} \cup \underbrace{\{x \in \Gamma : x.\gamma \in K,\ x.\gamma.s \notin K\}}_{A_s . \gamma^{-1}}$

we have $\forall s \in S \quad |A_{\gamma.s}| \leq |A_\gamma| + |\partial S|$

By induction, if $\gamma = s_1 . \cdots . s_n$, we obtain $\underline{|A_\gamma| \leq n\,|\partial S|}$.

Now, pick $\gamma \sim \mathrm{Uniform}(B_n)$ where $n = R(2|K|)$.

Since $|B_n| \geq 2|K|$, we have $\forall x \in K$ $\mathbb{P}[x.\gamma \in K] = \frac{|B_n(x) \cap K|}{|B_n|} \leq \frac{|K|}{|B_n|} \leq \frac{1}{2}$.

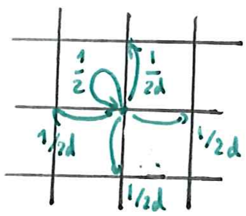


Furthermore $n.|\partial K| \geq \mathbb{E}[|A_\gamma|] = \mathbb{E}\left[\sum_{x \in K} 1\!\!1_{x.\gamma \notin K}\right] = \sum_{x \in K} \underbrace{\mathbb{P}[x.\gamma \notin K]}_{\geq \frac{1}{2}}$.

Therefore $\frac{|\partial K|}{|K|} \geq \frac{1}{2n} = \frac{1}{2R(2|K|)}$. ■

3 LAZY RANDOM WALK

Def: The lazy random walk (lazy RW) on G starting at x is the homogeneous Markov chain $(Y_m)_{m \geq 0}$ with
- state space V
- initial state $Y_0 = x$
- transition probabilities $\boxed{q(x,y) = \begin{cases} \frac{1}{2} & \text{if } x = y \\ \frac{1}{2} p(x,y) & \text{if } x \neq y. \end{cases}}$



" a slower version of the SRW "

main advantage: the lazy RW is always aperiodic.

Notation: • $Q := \frac{1}{2}(I + P)$ operator associated to the lazy RW.

• $q_m(x,y) = \mathbb{P}[Y_m = y \mid Y_0 = x]$ the m-step transition probabilities

Rk: $q_m(x,y) = \langle Q^m 1\!\!1_x, 1\!\!1_y \rangle = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} p_k(x,y)$.

We will use the following properties of the lazy RW:

Properties:

(i) $\forall\, n \geq 0 \quad p_{2n}(0,0) \leq 2\, q_{2n}(0,0)$. [comparison with SRW]

(ii) $\forall\, m,n \geq 0 \quad q_{m+n}(x,z) = \sum_{y \in V} q_m(x,y)\, q_n(y,z)$ [C-K]

Proof: (ii) is a standard property of Markov chains.

(i) First observe that $(p_{2k}(0,0))$ is decreasing in k.

Indeed $p_{2k+2}(0,0) = \sum_y p_2(0,y)\, p_{2k}(y,0) \leq p_{2k}(0,0) \underbrace{\sum_y p_2(0,y)}_{=1}$.

Then $q_{2n}(0,0) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2k} p_{2k}(0,0)$

$$\geq p_{2n}(0,0) \underbrace{\frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2k}}_{=\frac{1}{2}} \qquad \blacksquare$$

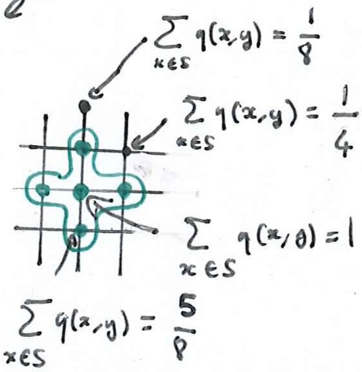
Rk: By (i), we have (Lazy RW transient) $\Rightarrow$ (SRW transient).
In fact, it is easy to see that it is an equivalence (see exercices).

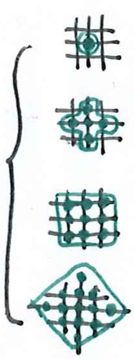
4. EVOLVING SETS.

Def: For $S \subset V$ finite and $u \in [0,1]$ we define the set

$$F(S,u) = \{ y \in V : \sum_{x \in S} q(x,y) \geq u \}$$

Example on $\mathbb{Z}^2$

$S =$ 

$F(S,u) =$ 

- if $u > \frac{5}{8}$
- if $\frac{1}{4} < u \leq \frac{5}{8}$
- if $\frac{1}{8} < u \leq \frac{1}{4}$
- if $0 \leq u < \frac{1}{8}$

Rk: If $u \leq \frac{1}{2}$ then $F(S,u) \supset S$ and $F(S,u) \setminus S \subset \underbrace{\{y \in S^c : \exists x \in S, x \sim y\}}_{}$ "outer boundary"

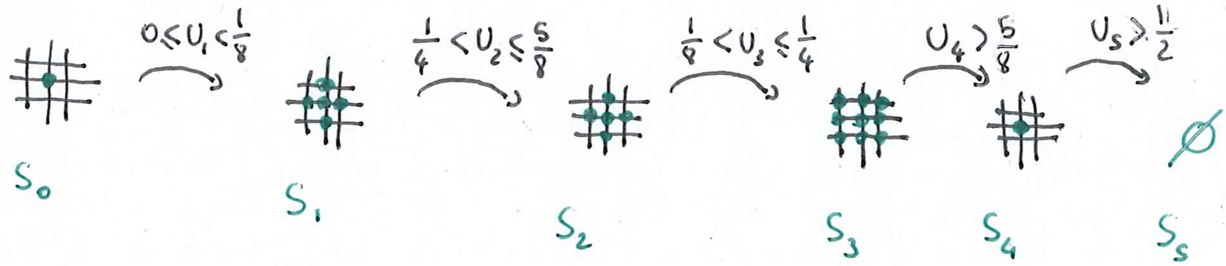
If $u \geq \frac{1}{2}$ then $F(S,u) \subset S$ and $S \setminus F(S,u) \subset \underbrace{\{x \in S : \exists y \in S^c, x \sim y\}}_{\text{"inner boundary"}}$

Def: Let $U_1, U_2, \dots$ be a sequence of iid uniform variables in $[0,1]$.

Define the Markov Chain $(S_n)_{n \in \mathbb{N}}$ by setting

- $S_0 = \{0\}$
- $S_{n+1} = F(S_n, U_{n+1})$.

Example on $\mathbb{Z}^2$



Rk: $\emptyset$ is an absorbing state.

Properties:

(i) For every n we have $q_n(0,x) = \mathbb{P}[x \in S_n]$.

(ii) $|S_n|$ is a martingale (for the filtration generated by $U_1, \dots, U_n$)

Remark: We have $q_n(0,0) = \mathbb{P}[S_n \neq \emptyset]$ and $q_n(0,0) \xrightarrow[n\to\infty]{} 0$ (exercice)

Therefore $S_n \to \emptyset$ a.s.

Proof: (i) By induction on n. First, we have $q_0(0,x) = \mathbb{1}_{x=0} = \mathbb{P}[S_0 \ni x]$

Then for $n \geq 0$, we have

$$q_{n+1}(0,x) = \sum_{y \in V} q_n(0,y)\, q(y,x) \overset{(IH)}{=} \sum_{y \in V} \mathbb{P}[y \in S_n]\, q(y,x)$$
$$= \mathbb{E}\Big[\sum_{y \in S_n} q(y,x)\Big].$$

Since $\sum_{y \in S_m} q(y,x) = \mathbb{P}\left[U_{m+1} \leq \sum_{y \in S_m} q(y,x) \mid S_m \right]$

$$= \mathbb{P}\left[x \in S_{m+1} \mid S_m \right],$$

we finally get $q_{m+1}(0,x) = \mathbb{P}\left[x \in S_{m+1} \right]$

(ii) $E\left[|S_{m+1}| \mid S_m \right] = \sum_{y \in V} \mathbb{P}\left[y \in S_{m+1} \mid S_m \right]$

$$= \sum_{y \in V} \sum_{x \in S_m} q(x,y)$$

$$= \sum_{x \in S_m} \underbrace{\sum_{y \in V} q(x,y)}_{=1} = |S_m|$$

5. TRANSIENCE FOR GRAPHS OF AT LEAST CUBIC GROWTH.

In this section we prove the following theorem.

Theorem:

Let φ be the expansion profile of G. Define $\varepsilon_m > 0$ by

$$\int_4^{4/\varepsilon_m} \frac{8d^2 \cdot du}{u\, \varphi(u)} = m .$$

Then we have $\boxed{q_{2m}(0,0) \leq \varepsilon_m}$.

Before proving let us see how it implies (ii) in the main theorem of the chapter. Assume G has at least cubic growth. The theorem in Section 2 implies that $\exists\, c > 0$ s.t.

$$\forall u \geq 1 \quad \varphi(u) \geq \frac{c}{u^{1/3}}$$

and the theorem above concludes that $\exists\, c' > 0$ s.t.

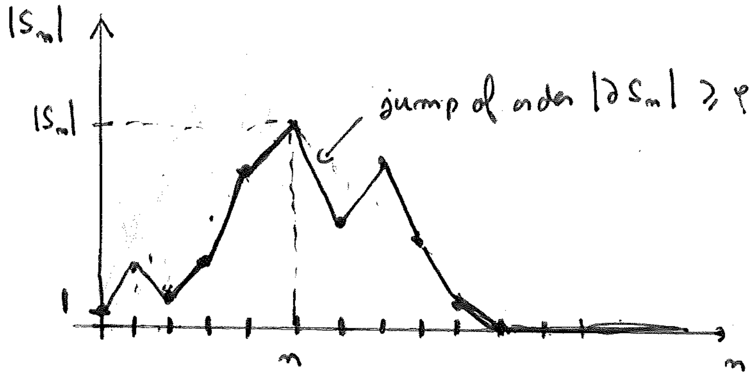
$$\forall m \geq 1 \quad q_{2m}(0,0) \leq \frac{c'}{m^{3/2}} .$$

Therefore $\sum_{m \geq 1} p_m(0,0) < \infty$ and the SRW is transient.

Idea of proof:

We want to bound $p_n(0,0) = \mathbb{P}[|S_n| = 0]$ where $|S_n|$ is martingale satisfying $|S_{n+1}| = |S_n| + Z_{n+1}$ where Z_n is a centered r.v. of order $\sqrt{E[Z_{n+1}^2]} \simeq |\partial S_n| \geq \varphi(|S_n|) \times S_n$. Roughly, $|S_n|$ is "faster" than the martingale M_n defined by $\begin{cases} M_0 = 1 \\ M_{n+1} = M_n + \varepsilon_{n+1}\, \phi(M_n) M_n \end{cases}$

where ε_n iid $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}[\varepsilon_i = -1] = \frac{1}{2}$.



↳ "reduction to a 1-D martingale problem". In order to get an estimate at which speed S_n gets absorbed at 0, we rather consider the supermartingale $\sqrt{|S_n|}$ which is "drifted" toward 0. ($\sqrt{|S_n|}$ is a super-martingale by Jensen inequality).

Introduce $\delta_n := E\left[\sqrt{|S_n|}\right]$.

Rk: $\delta_n \leq E[|S_n|]^{\frac{1}{2}} = 1$.

Lemma 1: For all $n \geq 0$ we have

$$\boxed{q_{2n}(0,0) \leq \delta_n^2}$$

Proof: Using Chapman-Kolmogorov for the lazy RW and reversibility, we find

$$q_{2n}(0,0) = \sum_{x \in V} q_n(0,x)^2 = \sum_{x \in V} \mathbb{P}[x \in S_n]^2$$

Introduce Σ_m, an independent copy of S_m.

$$
\begin{aligned}
q_{2m}(0,0) &= \sum_{x \in V} \mathbb{P}[x \in S_m, x \in \Sigma_m] \\
&= \mathbb{E}\Big[\sum_{x \in V} 1\!\!1_{x \in S_m} 1\!\!1_{x \in \Sigma_m}\Big] \\
&= \mathbb{E}\big[|S_m \cap \Sigma_m|\big] \\
&\leq \mathbb{E}\big[|S_m| \wedge |\Sigma_m|\big] \\
&\leq \mathbb{E}\Big[\sqrt{|S_m||\Sigma_m|}\Big] = \delta_m^2
\end{aligned}
$$ ■

Lemma 2.

For every S s.t. $\mathbb{P}[S_m = S] > 0$ and $S \neq \emptyset$, we have

- $\mathbb{E}\big[|S_{m+1}| \,\big|\, S_m = S, U_{m+1} < \frac{1}{2}\big] = |S| + \frac{1}{d}|\partial S|$, and
- $\mathbb{E}\big[|S_{m+1}| \,\big|\, S_m = S, U_{m+1} \geq \frac{1}{2}\big] = |S| - \frac{1}{d}|\partial S|$.

Proof: Since $(|S_m|)$ is a martingale, the second item follows from the first one.

For $y \in V$ we have

$$
\mathbb{P}\big[y \in S_{m+1} \,|\, S_m = S, U_{m+1} < \tfrac{1}{2}\big] = \mathbb{P}\Big[U_{m+1} \leq \sum_{x \in S} q(x,y) \,\Big|\, U_{m+1} < \tfrac{1}{2}\Big]
$$

$$
= \begin{cases} 1 & \text{if } y \in S \quad \text{(by laziness)} \\ \underbrace{2\sum_{x \in S} q(x,y)}_{= \frac{1}{d}\sum_{x \in S} 1\!\!1_{x \sim y}} & \text{if } y \notin S \end{cases}
$$

Therefore,

$$E[\,|S_{n+1}|\mid S_n=S,\,U_{n+1}<\tfrac12] = |S| + \frac1d \sum_{y\notin S}\sum_{x\in S} \mathbb{1}_{x\sim y} = |S| + \frac1d|\partial S| \quad \blacksquare$$

<u>Lemma 3</u>: For every $n\geq 0$, we have -

$$E\left[\sqrt{|S_{n+1}|}\mid S_n\right] \leq \left(1-\psi(|S_n|)\right)\sqrt{|S_n|} \quad \text{where } \psi(u) := \frac{1}{8d^2}\varphi(u)^2$$

<u>Proof</u>: We will use the following elementary inequality.

$$\forall\, t\in[0,1] \quad \frac{\sqrt{1+t}}{2} + \frac{\sqrt{1-t}}{2} \leq 1 - \frac{t^2}{8}. \quad \text{(exercice)}.$$

Let $S\subset V$ s.t. $\mathbb{P}[S_n=S]>0$ and $S\neq\emptyset$

$$E\left[\sqrt{|S_{n+1}|}\mid S_n=S\right] = \frac12 E\left[\sqrt{|S_{n+1}|}\mid S_n=S,\,U_{n+1}<\tfrac12\right] + \frac12 E\left[\sqrt{|S_{n+1}|}\mid S_n=S,\,U_{n+1}\geq\tfrac12\right]$$

Jensen + Lemma 2

$$\leq \frac12\sqrt{|S|+\frac1d|\partial S|} + \frac12\sqrt{|S|-\frac1d|\partial S|}$$

$$= \sqrt{|S|}\left(\frac12\sqrt{1+\frac1d\frac{|\partial S|}{|S|}} + \frac12\sqrt{1-\frac1d\frac{|\partial S|}{|S|}}\right)$$

$$\leq \sqrt{|S|}\left(1-\frac{1}{8d^2}\left(\frac{|\partial S|}{|S|}\right)^2\right)$$

$$\leq \sqrt{|S|}\left(1-\underbrace{\frac{1}{8d^2}\varphi(|S|)^2}_{=\psi(|S|)}\right) \quad \blacksquare$$

Proof of the theorem.

$$\delta_{m+1} = \mathbb{E}\left[\sqrt{|S_{m+1}|}\right] \underset{\substack{\uparrow \\ \text{Lemma 3}}}{\leq} \mathbb{E}\left[\sqrt{|S_m|}\,(1-\psi(|S_m|))\right] = \delta_m - \mathbb{E}\Big[\underbrace{\sqrt{|S_m|}}_{\nearrow \text{ in } |S_m|}\ \underbrace{\psi(|S_m|)}_{\searrow \text{ in } |S_m|}\Big]$$

Observation: $1 = \mathbb{E}[|S_m|] = \sum_{S \subset V} |S|\, \mathbb{P}[S_m = S]$

We introduce a random variable $S_m^\bullet$ which is a size-biased version of S_m, define by

$$\mathbb{P}[S_m^\bullet = S] = |S|\, \mathbb{P}[S_m = S]$$

This way, we have for every f measurable

$$\mathbb{E}[f(S_m)\cdot|S_m|] = \mathbb{E}[f(S_m^\bullet)].$$

Using this new variable, we obtain $\delta_m = \mathbb{E}\left[\frac{1}{\sqrt{|S_m^\bullet|}}\right]$

$$\mathbb{E}\left[\sqrt{|S_m|}\,\psi(|S_m|)\right] = \mathbb{E}\left[\frac{1}{\sqrt{|S_m^\bullet|}}\,\psi(|S_m^\bullet|)\right]$$

$$\geq \mathbb{E}\Big[\frac{1}{\sqrt{|S_m^\bullet|}}\,\underbrace{\psi(|S_m^\bullet|)}_{\leq \frac{4}{\delta_m^2}}\ \mathbb{1}_{\frac{1}{\sqrt{|S_m^\bullet|}} \geq \frac{\delta_m}{2}}\Big]$$

$$\geq \psi\left(\frac{4}{\delta_m^2}\right) \mathbb{E}\left[\frac{1}{\sqrt{|S_m^\bullet|}}\ \mathbb{1}_{\frac{1}{\sqrt{|S_m^\bullet|}} \geq \frac{\delta_m}{2}}\right]$$

$$\geq \psi\left(\frac{4}{\delta_m^2}\right)\cdot \frac{1}{2}\,\delta_m .$$

For the last inequality we use that for a positive integrable random variable X, $\mathbb{E}[X\,\mathbb{1}_{X \geq \frac{1}{2}\mathbb{E}[X]}] = \mathbb{E}[X] - \mathbb{E}[X\,\mathbb{1}_{X < \frac{1}{2}\mathbb{E}[X]}] \geq \frac{1}{2}\mathbb{E}[X]$.

Finally, we obtain $\forall m \geq 0$

$$\delta_{m+1} \leq \delta_m \left(1 - \frac{1}{2}\psi\left(\frac{4}{\delta_m^2}\right)\right).$$

This implies

$$\frac{\delta_{m+1}}{\delta_m} \leq e^{-\frac{1}{2}\psi\left(\frac{4}{\delta_m^2}\right)}$$

ie $$-\log(\delta_{m+1}) + \log(\delta_m) \geq \frac{1}{2}\,\psi\left(\frac{4}{\delta_m^2}\right)$$

which gives (using $\delta_{k+1} \leq \delta_k$ and ψ is decreasing) for every k

$$\int_{\delta_{k+1}}^{\delta_k} \frac{2\,dt}{t\,\psi(4/t^2)} \geq \frac{2}{\psi\left(4/\delta_k^2\right)} \int_{\delta_{k+1}}^{\delta_k} \frac{dt}{t} \geq 1$$

Summing over $k = 0, \dots, m-1$, we get.

$$\int_{\delta_m}^{\delta_0} \frac{2\,dt}{t\,\psi(4/t^2)} \geq m$$

By using the change of variable $u = 4/t^2$, we finally get

$$\int_4^{4/\delta_m^2} \frac{du}{u\,\psi(u)} \geq m = \int_4^{4/\varepsilon_m} \frac{du}{u\,\psi(u)}$$

Hence $\delta_m^2 \leq \varepsilon_m$ and Lemma 1 concludes that

$$q_{2m}(0,0) \leq \delta_m^2 \leq \varepsilon_m \qquad \blacksquare$$

– CHAPTER 4 –
– SPEED AND LIOUVILLE PROPERTY –

Ref: [LYONS - PERES, Chap. 14] [Ariel YADIN, lecture notes]

$G=(V,E)$ transitive, loc. finite, connected, infinite graph.

degree d, fixed origin $o \in V$

volume growth exponent v $\left(|B_n| = e^{vn+o(n)}\right)$

spectral radius ρ $\left(p_{2n}(o,o) = \rho^{2n+o(n)}\right)$

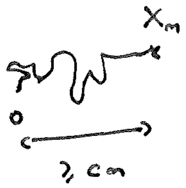
In this chapter we are interested in the asymptotic behaviour of $\frac{E(|X_n|)}{n}$.

We have already seen:

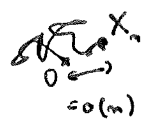
- If G has subexponential volume growth, then $\lim \frac{E(|X_n|)}{n} = 0$ "zero speed"
- If G is non amenable then $p_n(o,x) \leq \rho^n$ with $\rho < 1$

$\rightarrow \mathbb{P}_o[X_n \in B_{an}] \leq |B_{an}|\, \rho^n = e^{avn+o(n)} \rho^n \rightarrow 0$ if $e^{av}\rho < 1$,

this implies $\liminf \frac{|X_n|}{n} \geq \frac{v}{\log(1/\rho)} > 0$ "positive speed".



positive speed.



zero speed.

What happens for amenable graphs with exponential growth?

In this chapter, we will present the deep theorem of Kaimanovich and Vershik that gives several characterizations of positive speed (in terms of entropy / harmonic functions / tail σ-algebra).

1. SPEED OF THE S.R.W.

Write $|x| = d(o,x)$ for every $x \in V$.

We have $E_o[|X_{m+n}|] \leq E_o[|X_m|] + \underbrace{E_o[d(X_m, X_{m+n})]}_{= E_o[|X_n|]}$

(by Markov property + invariance)

By Fekete's lemma $\frac{E_o[|X_n|]}{n}$ converges in $[0,\infty)$.

Def: The speed of the SRW is defined by

$$\boxed{\ell = \lim_{n\to\infty} \frac{E_o[|X_n|]}{n} = \inf_{n \geq 1} \frac{E_o[|X_n|]}{n}.}$$

Ex:
- $\ell = 0$ if G has subexponential growth
- $\ell \geq \frac{v}{\log(1/\rho)}$ if G is non amenable.
- $\ell = \frac{d-2}{d}$ if G is a d-regular tree.

Proposition: We have

$$\boxed{\frac{|X_n|}{n} \xrightarrow[n\to\infty]{} \ell \quad \text{a.s.}}$$

Proof: Admit. Follows from an application of Kingman's subadditive theorem (see thm 14.10 in [LYONS-PERES])

2. ENTROPY.

Def: Let $\mathcal{X}$ be a finite set and X a r.v. with values in $\mathcal{X}$.
The Shannon entropy of X is defined by

$$\boxed{H(X) = -\sum_{x \in \mathcal{X}} \mathbb{P}[X=x] \log(\mathbb{P}[X=x]).}$$

Ref: See . Yadin "Harmonic functions on groups", Sect. 4.3.
. Cover and Thomas "Elements of information theory".
. blog "Yoo Box" $\to$ see "the town".

Intuition: $H(X) \simeq$ "expected number of bits needed to encode X"

$\simeq \log$ (" size of the set where X lives ")

Properties: • $H(X) \leq \log(|\mathcal{X}|)$ with equality iff $X \sim \text{Uniform}(\mathcal{X})$

• If X r.v. in $\mathcal{X}$ an Y r.v. in $\mathcal{Y}$ (with $|\mathcal{X}|<\infty$ and $|\mathcal{Y}|<\infty$)

$H(X,Y) \leq H(X) + H(Y)$ with equality iff X, Y indep.

$H(X) \leq H(X,Y)$ with equality iff $Y \in \sigma(X)$.

Entropy of the SRW

$$\boxed{\text{Define } H(X_n) = -\sum_{y \in V} \mathbb{P}_x[X_n = y] \log(\mathbb{P}_x[X_n = y]).}$$

Rk: Independent of the starting point x (by invariance).

Prop: $\boxed{\text{For } m, n \geq 0 \text{, we have } H(X_{m+n}) \leq H(X_m) + H(X_n).}$

Proof: $H(X_{m+n}) \leq H(X_m, X_{m+n})$

$$= -\sum_{x\in V} \mathbb{P}_0[X_m = x] \sum_{y \in V} \mathbb{P}_0[X_{m+n} = y \mid X_m = x] \times \log\left(\mathbb{P}_0[X_{m+n} = y, X_m = x]\right)$$

$$= -\sum_{x\in V} \mathbb{P}_0[X_m = x] \underbrace{\sum_{y\in V} \mathbb{P}_0[X_{m+n} = y \mid X_m = x] \log\left(\mathbb{P}_0[X_{m+n} = y \mid X_m = x]\right)}_{\overset{\text{(Markov)}}{=} H(X_n)}$$

$$- \sum_{x\in V} \mathbb{P}_0[X_m = x] \underbrace{\sum_{y\in V} \mathbb{P}_0[X_{m+n} = y \mid X_m = x]}_{=1} \log\left(\mathbb{P}_0[X_m = x]\right)$$

$$= H(X_n) + H(X_m) \qquad \blacksquare$$

As a consequence, by Fekete's lemma, $\frac{H(X_n)}{n}$ converges in $[0, \infty)$.

Def: The <u>Avez entropy</u> of the SRW is defined by

$$\boxed{h = \lim_{n\to\infty} \frac{H(X_n)}{n}.}$$

Intuition: "If $h > 0$, X_n lives in set of size $\simeq e^{hn}$"

Thm: For every $n \geq 0$, we have

(i) $H(X_n) \leq \log n + \log(d)\, E[|X_n|]$,

(ii) $n^2 \log\left(\frac{1}{\rho}\right) + E[|X_n|^2] \leq n\left(H(X_n) + \log 2\right)$.

Corollary: $\boxed{\rho > 0 \text{ if and only if } h > 0.}$

Proof $\boxed{\Leftarrow}$ follows from (i)

$\boxed{\Rightarrow}$ follows from $E[|X_n|]^2 \leq E[|X_n|^2]$ and (ii).

Proof of the theorem:

(i) $$H(X_m) = -\sum_x \mathbb{P}_0[X_m = x] \log(\mathbb{P}_0[X_m = x])$$

$$= \underbrace{-\sum_k \mathbb{P}_0[|X_m| = k] \log(\mathbb{P}_0[|X_m| = k])}_{\leq \log m}$$

$$+ \sum_k \mathbb{P}_0[|X_m| = k] \underbrace{-\sum_x \mathbb{P}_0[X_m = x \,|\, |X_m| = k] \log(\mathbb{P}_0[X_m = x \,|\, |X_m| = k])}_{\leq \log |B_k \setminus B_{k-1}|}$$

$$\leq \log m + \mathbb{E}[\log |B_{X_m}|]$$

$$\leq \log m + (\log d)\, \mathbb{E}[|X_m|]$$

(ii) Recall Varopoulos - Carne bound:

$$\mathbb{P}_0[X_m = x] \leq 2\rho^m \exp\left(-\frac{|x|^2}{2m}\right)$$

This implies

$$H(X_m) \geq \sum_x \mathbb{P}_0[X_m = x]\left(-\log 2 - m \log \rho + \frac{|x|^2}{2m}\right)$$

$$= -\log 2 - m \log \rho + \frac{1}{m} \mathbb{E}_0[|X_m|^2]$$

Therefore

$$\mathbb{E}[|X_m|^2] \leq m\left(H(X_m) + \log 2\right) - m^2 \log\left(\frac{1}{\rho}\right) \blacksquare$$

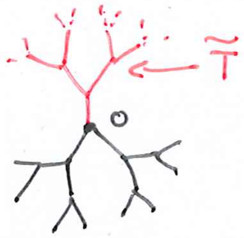
<u>Def</u>: Let $(X_n)_{n\geq 0}$ be a SRW starting at 0.

Define $\boxed{\mathcal{T} = \bigcap_{m\geq 0} \sigma(X_m, X_{m+1}, \dots)}$ "tail σ-algebra".

<u>Def</u>: We say that $\mathcal{T}$ is <u>trivial</u> if $\forall A \in \mathcal{T}$ $\mathbb{P}_0(A) \in \{0,1\}$.

<u>Rk</u>: If $G = \mathbb{T}_3$ 3-regular tree, consider $\tilde{T} \subset \mathbb{T}_d$ defined

by the diagram:



Let $A = \{X_n \text{ escapes in } \tilde{T}\} = \bigcup_{n_0 \geq 0} \{\forall k \geq n_0\ X_k \in \tilde{T}\}$

We can check that $A \in \mathcal{T}$ and $\mathbb{P}_0[A] = \frac{1}{3}$.

$\rightarrow$ "$\mathcal{T}$ is not trivial for the tree".

<u>Thm</u>: $\boxed{h > 0 \text{ iff } \mathcal{T} \text{ is not trivial.}}$

<u>Proof</u>: Admit. see [LYONS-PERES], Thm 14.7.

3. HARMONIC FUNCTIONS.

<u>Def</u>: We define the operator $\Delta : \mathbb{R}^V \longrightarrow \mathbb{R}^V$ by

$$\forall f \in \mathbb{R}^V \quad \forall x \in V \quad (\Delta f)(x) = \frac{1}{d} \sum_{y\sim x} f(y) - f(x).$$

<u>Rk</u>: The operator P can also be defined on $\mathbb{R}^V$ and we have

$$\Delta = P - Id.$$

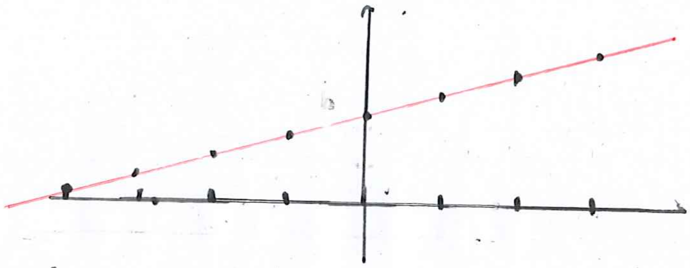
Def: $f : V \to \mathbb{R}$ is <u>harmonic</u> if $\Delta f = 0$.

Rk: f is harmonic iff $\forall x \quad f(x) = \frac{1}{d} \sum_{y \sim x} f(y)$.

Exples.
- constant functions.
- $G = \mathbb{Z}$: $\quad (f : \mathbb{Z} \to \mathbb{R}$ harmonic$)$

$$\iff \forall n \in \mathbb{Z} \quad f(n+1) - f(n) = f(n) - f(n-1)$$

$$\iff \exists a, b \in \mathbb{R} \text{ s.t. } \forall n \quad f(n) = an + b$$



- $G = \mathbb{Z}^2$: $\quad h((x,y)) = x^2 - y^2$.
- $G = \mathbb{T}_3$: $\quad h(x) = \mathbb{P}_x[(X_n) \text{ escapes in } \tilde{T}]$

(exercice : check that h is harmonic and compute $h(x)$)

<u>Connection with the SRW</u>:

Let $h \in \mathbb{R}^V$. The following are equivalent ((X_n) S.R.W from x)

(i) h is harmonic

(ii) $Ph = h$

(iii) $(h(X_n))_{n \geq 0}$ is a martingale (w.r.t to $\sigma(X_0, \dots, X_n)$).

Pf: (ii) $\Rightarrow$ (iii) $\quad E_x[h(X_{n+1}) \mid X_0, \dots, X_n] = \sum_{y \in V} p(X_n, y) h(y)$

$$= (Ph)(X_n) = X_n$$

(iii) $\Rightarrow$ (ii) : Assume $(h(X_n))_{n\geq 0}$ martingale.

Let $y \in V$. By irreducibility there exists m s.t. $\mathbb{P}_x[X_m = y] > 0$.

$$Ph(y) = E_y[h(X_1)] \overset{\text{Markov.}}{=} \mathbb{E}_x[h(X_{m+1}) \mid X_m = y]$$

$$= h(y)$$

■

4) LIOUVILLE PROPERTY. (LP)

Def: We say that G has the Liouville property if every bounded harmonic function is constant.

Ex: $\mathbb{Z}$ has LP. $\mathbb{T}_3$ does not have LP ($h(x) = \mathbb{P}_x[(X_n)$ escape in $\check{T}]$)

Prop: Assume that for all x, y neighbours, $\exists (X_n)_{n\geq 0}, (Y_n)_{n\geq 0}$ s.t.

- $(X_n)_{n\geq 0}$ SRW from x and $(Y_n)_{n\geq 0}$ SRW from y
- $\mathbb{P}[\exists n_0 : \forall n \geq n_0 \; X_n = Y_{n+1}] = 1$

Then G has LP.

Rk: A characterisation exists (see [YADIN], sect. 4.6)

Proof: Let h be a bounded harmonic function.

$$|h(x) - h(y)| = |\mathbb{E}[h(X_n)] - \mathbb{E}[h(Y_n)]|$$
$$\leq \mathbb{E}[|h(X_n) - h(Y_n)|]$$
$$\leq 2\|h\|_\infty \mathbb{P}[X_n \neq Y_n] \xrightarrow[n\to\infty]{} 0$$

■

Application: $\mathbb{Z}^d$ has LP.

> **Prop:** $LL(G)$ has LP iff G is recurrent.

Proof: (exercise)

$\boxed{\Leftarrow}$ use a coupling method similar to the proposition.

Couple two lazy RW (X_m, ξ_m) and (Y_m, ζ_m)

X_m: position in G; ξ_m: lamp configuration

First couple X_m and Y_m, and then match the lamps.

$\boxed{\Rightarrow}$ Consider $h(x) = \mathbb{P}_x($ the lamp at 0 is on for n sufficiently large$)$.

Theorem: [Kaimanovitch - Vershik].

> The following are equivalent.
>
> (i) $\rho > 0$ "positive speed" — "Random walk"
>
> (ii) $h > 0$ "positive Avez entropy" — "information theory"
>
> (iii) $\mathcal{T}$ is non-trivial. — "measure theory"
>
> (iv) G does not have LP. — "non trivial boundary"

Proof: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) already seen.

(iv) $\Rightarrow$ (iii) Assume $\mathcal{T}$ is trivial. Let h be a bounded HF.

By the martingale convergence theorem $h(X_n) \xrightarrow{a.s.} Y$ ((X_n) SRW from 0)

Since $\mathcal{T}$ is trivial, $Y = c$ constant.

Fix $x \in V$ and choose m s.t. $\mathbb{P}_0[X_m = x] > 0$

$$E[\mathbb{1}_{X_m = x} \cdot h(X_n)] \longrightarrow E[\mathbb{1}_{X_m = x}\, c] = c\, P[X_m = x]$$

$$\| $$

$$E[h[X_n] \mid X_m = x]\, P[X_m = x] = h(x)\, \mathbb{P}[X_m = x]$$

(iii) $\Rightarrow$ (iv) admit. (non trivial).