PART 2:

PERCOLATION ON TRANSITIVE GRAPHS

CHAPTER 1 : DEFINITIONS AND FUNDAMENTAL PROPERTIES.

Ref: [GRIMMETT, Percolation, Chap. 1-2] [LYONS - PERES, Chap. 7]

$G = (V, E)$ transitive, loc. finite, connected, infinite graph. degree d.

1. DEFINITIONS

1.1 Percolation configurations

An element $\omega \in \{0,1\}^E$ is called a (bond) percolation configuration.

Def: Let $\omega \in \{0,1\}^E$.

- An edge e is said to be open if $\omega(e) = 1$.
- A path $\gamma = (\gamma_0, \dots, \gamma_\ell)$ is open if $\forall i < \ell$ $\omega(\gamma_i \gamma_{i+1}) = 1$.
- A cluster is a connected component of $G_\omega = (V, E_\omega)$, where
$$E_\omega = \{ e \in E : \omega(e) = 1 \}.$$

Not: $C_x(\omega)$: cluster of x.
$N(\omega)$: number of infinite clusters.

1.2. Percolation measure. $p \in [0,1]$

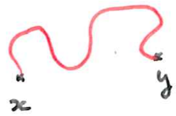
We consider the probability space $(\{0,1\}^E, \mathcal{F}, \mathbb{P}_p)$, where

- $\mathcal{F}$ is the product σ-algebra.
- $\mathbb{P}_p = \prod_{e \in E} (p\delta_1 + (1-p)\delta_0)$ percolation measure with density p.

Rk: If $X \sim \mathbb{P}_p$ then $(X(e))_{e \in E}$ iid Bernoulli(p).

Some events : $x, y \in V$

- $(x \leftrightarrow y) = \{ \exists$ open path from x to $y \}$.



- $(x \leftrightarrow \infty) = \{ x$ belongs to an infinite cluster$\}$.

Ex: Check that $x \leftrightarrow y$ and $x \leftrightarrow \infty$ are measurable.
Check that $\omega \mapsto N(\omega)$ is measurable.

2. MONOTONICITY.

Question: How do we prove that $P_p[x \leftrightarrow y]$ is non-decreasing in p?

Def: Equip $\{0,1\}^E$ with the product ordering :

$$\omega \leq \eta \iff \forall e \in E \ \ \omega(e) \leq \eta(e).$$

A function $f : \{0,1\}^E \to \mathbb{R}$ is increasing if

$$\omega \leq \eta \Rightarrow f(\omega) \leq f(\eta).$$

Ex $|C_x|$ is increasing. N is neither increasing or decreasing.

Def: An event A is increasing if 1_A is increasing.

Ex: $x \leftrightarrow y$, $x \leftrightarrow \infty$.

Prop: Let $f : \{0,1\}^E \to \mathbb{R}$ increasing bounded or ≥ 0. Then

$$p \mapsto E_p[f] \text{ is increasing}$$

Proof: Monotone coupling. Let $(U_e)_{e \in E}$ be iid uniform in $[0,1]$.

Define for every $p \in [0,1]$ $X_p(e) = 1_{U_e \leq p}$.

We have $X_p \sim P_p$ and $p \leq p' \Rightarrow X_p \leq X_{p'}$ a.s.

Hence $\underbrace{E[f(X_p)]}_{= E_p[f]} \leq \underbrace{E[f(X_{p'})]}_{= E_{p'}[f]}$ ■

Appli: $P_p[x \leftrightarrow y]$, $P_p[x \leftrightarrow \infty]$ increasing in p.

3 Positive Association (Harris-FKG inequality.)

Thm: Let $f, g : \{0,1\}^E \to \mathbb{R}$ increasing bounded. Then

$$\mathbb{E}_p[f \times g] \geq \mathbb{E}_p[f]\, \mathbb{E}_p[g]$$

Let A, B be two increasing events. Then

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\, \mathbb{P}_p[B].$$

Proof: See [Grimmett, p.34] or [Lyons-Peres, Sect. 5.8]

Rk: The second inequality can be rewritten as

$$\mathbb{P}_p[A \mid B] \geq \mathbb{P}_p[A]$$

"knowing that B occurs, A has more chance to occur"

Appl: $\mathbb{P}_p[x \leftrightarrow z] \geq \mathbb{P}_p[x \leftrightarrow y]\, \mathbb{P}_p[y \leftrightarrow z]$.

4. Insertion / Deletion Tolerance

Def: For $e \in E$ and $\omega \in \{0,1\}$. Define $\Pi^e \omega$ and $\Pi_e \omega$ in $\{0,1\}^E$ by

$$(\Pi^e \omega)(f) = \begin{cases} \omega(f) & f \neq e \\ 1 & f = e. \end{cases} \qquad \text{"put } e \text{ open"}$$

$$(\Pi_e \omega)(f) = \begin{cases} \omega(f) & f \neq e. \\ 0 & f = e \end{cases} \qquad \text{"put } e \text{ closed"}$$

Prop: Let $0 < p < 1$. Then $\exists c > 0$ s.t. the following holds.
$\forall A \in \mathcal{F}$

$$\mathbb{P}_p[\Pi^e A] \geq c\, \mathbb{P}_p[A] \qquad \text{"insertion tolerance"}$$

$$\mathbb{P}_p[\Pi_e A] \geq c\, \mathbb{P}_p[A] \qquad \text{"deletion tolerance"}$$

Proof: Let $\tilde{A} = \{\omega : \Pi_e \omega \in A \text{ or } \Pi^e \omega \in A\}$ (it is measurable). Using that $\tilde{A}$ is independent of $\omega(e)$ and $\tilde{A} \supset A$, we find

$\mathbb{P}_p(\Pi^e A) = \mathbb{P}(\Pi^e A \mid \tilde{A})\, \mathbb{P}_p(\tilde{A}) = p\, \mathbb{P}_p(\tilde{A}) \geq p\, \mathbb{P}_p(A)$. Equivalently $\mathbb{P}(\Pi_e A) \geq (1-p)\mathbb{P}_p(A)$ ∎

Application Let $0<p<1$. Let $k \in \{1,2,\dots\}$. Then $\exists c>0$ s.t.

$$\mathbb{P}_p[N=1] \geq c\, \mathbb{P}_p[N=k].$$

Proof: Let $E_n = \{$ all the infinite clusters intersect $B_n\}$.

Since $\mathbb{P}[N=k] = \lim_{n\to\infty} \mathbb{P}_p[\{N=k\}\cap E_n]$ (because G is connected), we can pick n large enough such that

$$\mathbb{P}_p[\{N=k\}\cap E_n] \geq \frac{\mathbb{P}_p[N=k]}{2}.$$

the event $\{N=k\}\cap E_n$

By insertion tolerance we can "open" all the edges intersecting B_n. We obtain

$$\mathbb{P}_p[N=1] \geq c^{|E(B_n)|}\, \mathbb{P}_p[\{N=k\}\cap E_n]$$

$$\geq \frac{c^{|E(B_n)|}}{2}\, \mathbb{P}_p[N=k]. \qquad \blacksquare$$

5 INVARIANCE

Rk: $\Gamma = \mathrm{Aut}(G)$ acts on

- V : $\varphi\cdot x = \varphi(x)$
- E : $\varphi\cdot xy = \varphi(x)\varphi(y)$
- $\{0,1\}^E$: $(\varphi\cdot\omega)(e) = \omega(\varphi^{-1}\cdot e)$
- $\mathcal{F}$: $\varphi\cdot A = \{\varphi\cdot\omega,\ \omega\in A\}$.

Why do we take φ^{-1} in the action on the percolation configurations?

e open in $\omega \iff \varphi\cdot e$ open in $\varphi\cdot\omega$.

Ex If $A = x \longleftrightarrow y$ then $\varphi\cdot A = \varphi(x)\longleftrightarrow\varphi(y)$

Prop: For every $A \in \mathcal{F}$ and every $\varphi \in \mathrm{Aut}(G)$,

$$P_p(\varphi \cdot A) = P_p(A) \qquad \text{"} P_p \text{ is invariant".}$$

Proof: True for cylinder events. Conclude with monotone class theorem.

6. Mixing Property.

Not. For every $x \in V$, we consider $\varphi_x \in \mathrm{Aut}(G)$ s.t. $\varphi_x(0) = x$.

Prop [Mixing property] Let $A, B \in \mathcal{F}$. Then

$$\lim_{|x| \to \infty} P_p(A \cap \varphi_x \cdot B) = P_p(A) P_p(B).$$

Proof: Let $\varepsilon > 0$. Choose $A_\varepsilon, B_\varepsilon$ cylinder events s.t.

$$P_p(\underset{\substack{\uparrow \\ \text{"sym. difference"}}}{A \,\Delta\, A_\varepsilon}) \leq \varepsilon \quad \text{and} \quad P_p(B \,\Delta\, B_\varepsilon) \leq \varepsilon.$$

By independence, if $|x|$ is large enough, we have

$$P_p(A_\varepsilon \cap \varphi_x \cdot B_\varepsilon) = P_p(A_\varepsilon) P_p(\varphi_x \cdot B_\varepsilon)$$
$$= P_p(A_\varepsilon) P(B_\varepsilon).$$

Therefore, if $|x|$ large enough.

$$P_p(A \cap \varphi_x \cdot B) \leq P(A_\varepsilon \cap \varphi_x \cdot B_\varepsilon) + 2\varepsilon$$
$$= P_p(A_\varepsilon) P_p(B_\varepsilon) + 2\varepsilon$$
$$\leq P_p(A) P_p(B) + 4\varepsilon$$

The reverse inequality is proved the same way, and we finally get $|P_p(A \cap \varphi_x \cdot B) - P_p(A) P_p(B)| \leq 4\varepsilon$ ■

Application Fix $p \in [0,1)$ let $\theta = \mathbb{P}_p(0 \leftrightarrow \infty)$.

Prove that $X_n := \frac{1}{B_n} \sum_{x \in B_n} 1_{x \leftrightarrow \infty}$ converges to θ a.s. as $n \to \infty$.

Proof: exercise.

7. ERGODICITY:

Prop. Let A be an invariant event (i.e. $\forall \varphi \in \mathrm{Aut}(G)$ $\varphi . A = A$). Then

$$\mathbb{P}_p(A) \in \{0,1\}. \qquad \text{"}\mathbb{P}_p \text{ is ergodic"}.$$

Proof: By invariance of A, $\mathbb{P}_p(A) = \mathbb{P}_p(A \cap \varphi_x . A)$. Hence

$$\mathbb{P}_p(A) = \lim_{|x| \to \infty} \mathbb{P}_p(A \cap \varphi_x . A) \underset{\uparrow}{=} \mathbb{P}_p(A)^2 \qquad \blacksquare$$

"mixing"

Application: The number of infinite clusters is constant a.s., and there exists $k \in \{0,1,\infty\}$

$$\mathbb{P}_p[N = k] = 1.$$

Proof: $\{N = k\}$ is invariant and therefore $\mathbb{P}[N=k] \in \{0,1\}$.
Therefore $\exists\, k \in \{0,1,\dots\} \cup \{\infty\}$ s.t. $\mathbb{P}[N=k]=1$.
Since $\mathbb{P}_p[N=1) \geq c\, \mathbb{P}[N=k]$ if $1<k<\infty$, we cannot have $\mathbb{P}_p[N=k]=1$ for $1<k<\infty$. $\blacksquare$

CHAPTER 2

PHASE TRANSITION

$G=(V,E)$ transitive, loc. finite, connected, infinite graph degree d.

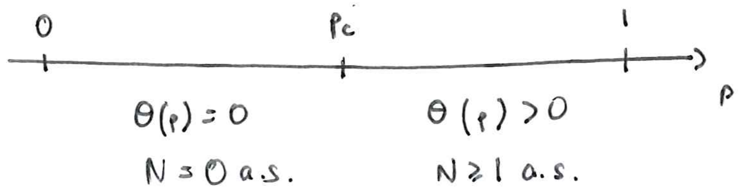
1. DEFINITION.

Not. $\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) = \mathbb{P}_p(|C_0| = \infty)$ (non decreasing in p)

$\theta_n(p) = \mathbb{P}_p(0 \leftrightarrow \partial_V B_n) = \mathbb{P}_p(\ \ )$ where $\partial_V B_n = B_n \setminus B_{n-1}$.

Rk: $\theta(p) = \lim_{n\to\infty} \theta_n(p)$ (decreasing limit).

> Def: The critical parameter for percolation is defined by
>
> $$p_c = p_c(G) = \sup\{p : \theta(p) = 0\}.$$



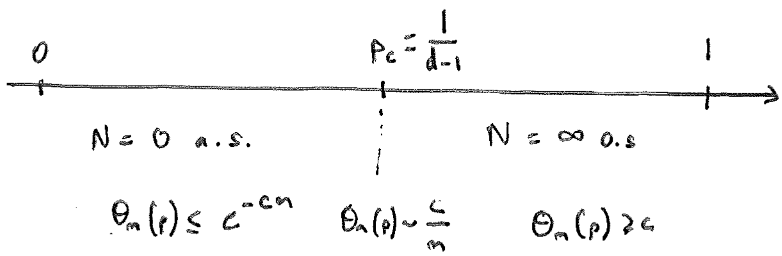
Rk: If $p<p_c$, $N=0$ a.s. If $p>p_c$ $N \geq 1$ a.s

2. Percolation on the tree $\mathbb{T}^d$

C_0 has the law of a Galton-Watson tree with offspring distribution X satisfying $E_p(X) = (d-1)p$ except at the first generation

$\boxed{\textbf{Thm.} \text{ The critical parameter for percolation on } \mathbb{T}^d \text{ is } \quad p_c(\mathbb{T}^d) = \frac{1}{d-1}.}$

Using the theory of branching processes, one can prove the following behaviors for percolation on $\mathbb{T}^d$.



3 LOWER BOUND ON p_c.

$\boxed{\textbf{Thm:} \text{ We have } \quad p_c \geq \frac{1}{d-1}.}$

__Proof__: Let $\Pi_n = \{$ paths of length n starting at $0.\}$.

For every $n \geq 1$

$$\begin{aligned}
\theta_n(p) &\leq \mathbb{P}_p\left(\exists \pi \in \Pi_n \text{ s.t. } \pi \text{ is open}\right) \\
&\leq \sum_{\pi \in \Pi_n} \underbrace{\mathbb{P}_p(\pi \text{ is open})}_{= p^n} \\
&= |\Pi_n|\, p^n \\
&\leq dp\left((d-1)p\right)^{n-1} \longrightarrow 0 \qquad \text{if } p < \frac{1}{d-1}
\end{aligned}$$

■

__Rk__: The proof shows that $\forall p < \frac{1}{d-1}$

$\theta_n(p)$ decays exponentially fast towards 0.

4. SHARPNESS OF THE PHASE TRANSITION.

Thm. $\forall p < p_c$ we have

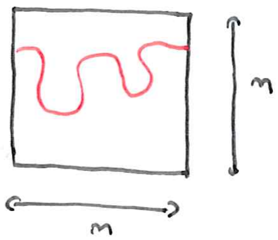
(i) $E_p(|C_o|) < \infty$

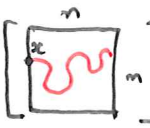
(ii) $\exists c > 0$ s.t. $\forall n \geq 1 \quad \theta_n(p) \leq e^{-cn}$.

Proof. admit.

Idea: when $p < p_c$, one can dominate the cluster C_o by a subcritical branching process.

Application. On $\mathbb{Z}^d$, $d \geq 2$, consider the box $\Lambda_n = \{0, n\}^d$.



If $p < p_c$ $\quad P_p(\exists \text{ path from left to right in } \Lambda_n) \leq \sum_{x \in \text{left side}} P_p\left[\right.$  $\left.\right]$

$\leq (n+1)^{d-1} \, e^{-cn}$

$\xrightarrow[n \to \infty]{} 0.$

5. UPPER BOUNDS ON p_c.

Def: We say that the phase transition is <u>non-trivial</u> if $p_c < 1$.

Rk: The phase transition is trivial if $G = \mathbb{Z}$ or $G = \mathbb{Z} \times H$, H finite.

Conjecture: The phase transition is non trivial if and only if G has at least quadratic growth. Ie.

$$(p_c(G) < 1) \iff (\exists c \text{ s.t. } \forall n \geq 1 \ B_n \geq c n^2).$$

Open question: What is $\sup\{p_c(G) : G \text{ transitive s.t. } p_c(G)<1\}$?

Rk: In order to give upper bounds on p_c, we have several possibilities

1. Prove that $\theta(p)>0$ for some $p<1$.
   → "difficult in general" this corresponds to showing the survival of some process.

2. Use sharpness of the phase transition: Prove that
   $E_p(|C_0|)=+\infty$ or $\frac{-\log \theta_n(p)}{n} \xrightarrow[n\to\infty]{} 0$ for some $p<1$.
   → "easier in general".

Thm. If G has exponential volume growth then the phase transition is non-trivial. More precisely, we have

$$p_c \leq e^{-v}.$$

Rk 1. If G is non amenable p_c can be bounded in terms of the isoperimetric constant

$$p_c \leq \frac{1}{1+\inf\limits_{n\geq 1}\frac{|\partial_V B_n|}{|B_n|}} \leq \frac{1}{1+\frac{\phi}{d}}.$$

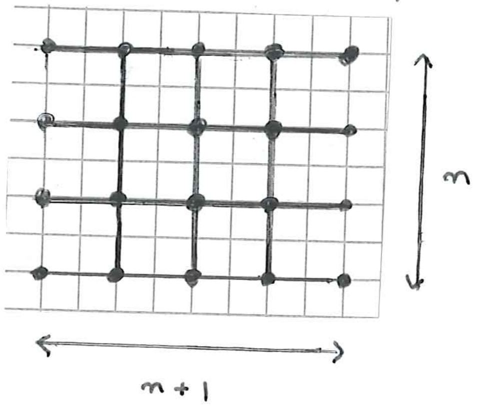
2. Together with the fact that $p_c \geq \frac{1}{d-1}$, the theorem above implies $p_c(\mathbb{T}^d)=\frac{1}{d-1}$ (in a complicated way).

Proof: $E_p(|C_0|) \geq E_p(|C_0\cap B_n|) = \sum\limits_{x\in B_n} \underbrace{P_p(0\leftrightarrow x)}_{\geq p^{|x|}} \geq \underbrace{|B_n|}_{e^{vn+o(n)}} p^n.$

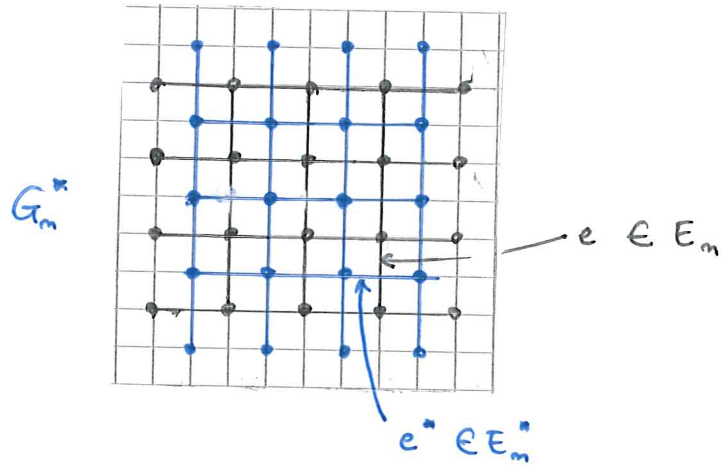
Hence $E_p(|C_0|)=+\infty$ if $e^{-v}<p<1$. This concludes $p_c\leq e^{-v}$. ∎

Thm: $p_c(\mathbb{Z}^2) \leq \frac{1}{2}$.

Proof. For $m \geq 1$ consider the graph $G_m = (V_m, E_m)$ defined by.

$G_m =$ 

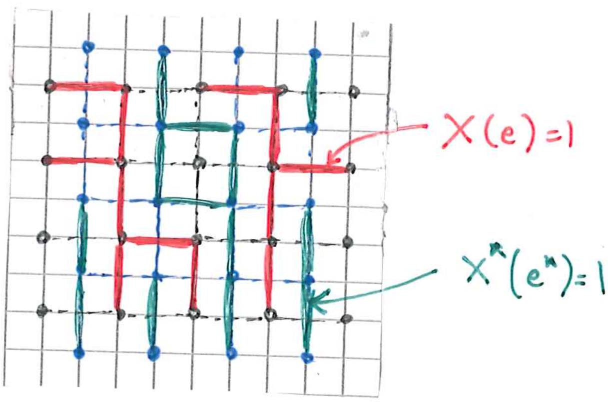
Let $G_m^* = (V_m^*, E_m^*)$ defined by the blue graph below



Notice that • an edge e of E_m "crosses" exactly one edge $e^* \in E_m^*$
• G_m and G_m^* are isomorphic.

Let $(X(e))_{e \in E_m}$ iid $\mathbb{P}(X(e)=1) = p$

Define $\forall\, e \in E_m$ $X^*(e^*) = 1 - X(e)$. In particular $\mathbb{P}(X^*(e^*)=1) = 1-p$.

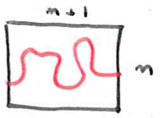
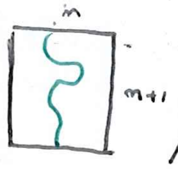


This way, we have realized a coupling between a percolation configuration with intensity p on G_n, and a percolation configuration with intensity $1-p$ on G_n^* in such a way that

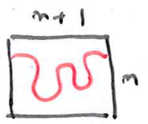
$$(\exists \text{ left right path in } X) \iff (\exists \text{ no top-down path in } X^*)$$

(we omit the proof of this combinatorial result.)

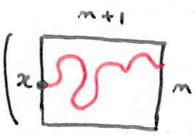
This implies

$\mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big) = 1$

If $p = \frac{1}{2}$ the two probabilities above are equal and we get

$P_{\frac{1}{2}}\Big($  $\Big) = \frac{1}{2}.$

Therefore:

$\frac{1}{2} \leq \sum_{\substack{x \text{ on the} \\ \text{left side}}} P_{\frac{1}{2}}\Big($  $\Big) \leq n\, \theta_n\left(\tfrac{1}{2}\right).$

This concludes

$$\theta_n\left(\tfrac{1}{2}\right) \geq \frac{1}{2n}.$$

In particular $\theta_n\left(\frac{1}{2}\right)$ does not decay exponentially, which concludes that $\frac{1}{2} \geq p_c$. ∎

Rk: Using BK inequality (see [Grimmett, Chap. 2]) and choosing x in the center of the box in the last step of the proof above, one can show $\theta_n\left(\frac{1}{2}\right) \geq \frac{1}{4\sqrt{n}}$.

Rk: For every $d \geq 2$. $\mathbb{Z}^2$ is a subgraph of $\mathbb{Z}^d$, hence

$$\frac{1}{d-1} \leq p_c(\mathbb{Z}^d) \leq p_c(\mathbb{Z}^2) \leq \frac{1}{2}.$$

"non trivial phase transition for percolation on $\mathbb{Z}^d$, $d \geq 2$".

Thm: If G has polynomial and at least quadratic growth, then $p_c(G) < 1$.

Proof sketch. 1. $p_c(\mathbb{Z}^2) \leq \frac{1}{2}$

2. If G is a Cayley graph of a nilpotent group with $|B_n| \geq cn^2$. Then G has $\mathbb{Z}^2$ as a subgraph. Therefore $p_c(G) \leq \frac{1}{2}$.

3. [Gromov - Trofimov] Assume G with at least quadratic growth, since G is transitive, $G=(V,E)$ is roughly-isometric to a Cayley graph $G'=(V',E')$ of a nilpotent group with $|B'_n| \geq cn^2$. I.E. $\exists\ \varphi : V \to V'$ and c constant s.t.

$$\forall x,y \in V \quad \frac{1}{c} d_G(x,y) - C \leq d_{G'}(\varphi(x),\varphi(y)) \leq C d_G(x,y) + C.$$

This can be used to conclude that $p_c(G) < 1$. ∎

CHAPTER 3 : PHASE TRANSITION FOR UNIQUENESS.

$G = (V, E)$ transitive loc. finite, connected, infinite graph. degree d.

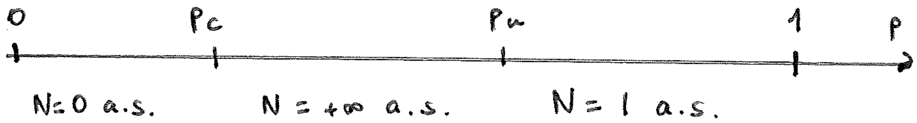
1. DEFINITION OF p_u.

Thm: Let $p_1 \leq p_2 \leq 1$. If $\mathbb{P}_{p_1}[N=1] = 1$, then $\mathbb{P}_{p_2}[N=1] = 1$.

Rk: $\{N=1\}$ is neither increasing nor decreasing. Hence, the theorem above does not follow from a simple monotonicity argument.

Proof: see [Lyons - Peres, Section 7.5]. Later we will prove the theorem in the case where G is unimodular.

Def: $\boxed{p_u = \inf \{p \geq 0 : \mathbb{P}_p[N=1] = 1\}}$.



Rk: For $G = \mathbb{T}_d$, $d \geq 3$, we have $p_u = 1$.

Def. We say that G has one end if for every $n \geq 1$, the graph induced by $V \setminus B_n$ is connected.

Ex: . $\mathbb{T}^d$, $d \geq 3$, $\mathbb{Z}$ do not have one end. ($\mathbb{Z}$ has two ends $\mathbb{T}^d$, $d \geq 3$ has infinitely many ends)

. $\mathbb{Z}^d$ has one end when $d \geq 2$.

Exercice: prove that $p_u = 1$ if G does not have one end.

Conjecture: $\boxed{(p_u < 1) \iff (G \text{ has one end}).}$

Rk: The conjecture is known for Cayley graphs of finitely presented, finitely generated groups.

One major conjecture is the following.

Conjecture: $\boxed{(p_c < p_u) \iff (G \text{ is non amenable}).}$

In the next section, we prove one implication.

2. UNIQUENESS FOR AMENABLE GRAPHS.

Thm: If G is amenable, then for every $p \in [0,1]$
$$\mathbb{P}_p[N = \infty] = 0.$$

Corollary: If G is amenable then $p_c = p_u$.

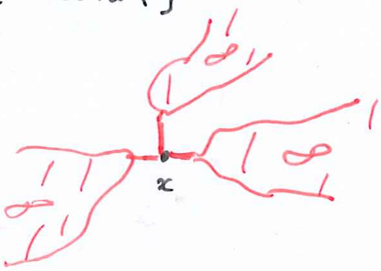
Lemma: Let (T, F) be a a finite tree (a finite connected graph with no cycle).
Let $N_1 = |\{x \in T : \deg(x) = 1\}|$, $N_{\geq 3} = |\{x \in T : \deg(x) \geq 3\}|$.
Then
$$N_1 \geq 2 + N_{\geq 3}.$$

Proof: We have $|T| = |F| - 1$ (by induction on $|T|$).
Write $N_2 = |\{x \in T : \deg(x) = 2\}|$.
By counting the edges of the trees in two different ways, we find $2|F| = \sum_{x \in T} \deg(x) \geq N_1 + 2N_2 + 3N_{\geq 3}$.
Since $2|F| = 2|T| + 2 = 2(N_1 + N_2 + N_{\geq 3}) + 2$, we obtain
$$N_1 \geq N_{\geq 3} + 2$$

Def: Let $\omega \in \{0,1\}^E$. A vertex $x \in V$ is called a trifurcation (in the configuration ω) if

- x has exactly 3 adjacent open edges.
- C_x splits into 3 disjoint infinite clusters if we close the edges adjacent to x.

Not: $T_x = \{x \text{ is a trifurcation}\}$



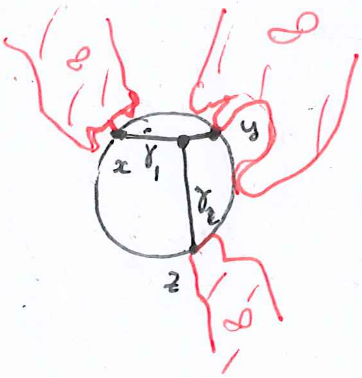
Lemma 2 Let $p \in [0,1]$. If $P_p[N=\infty] = 1$, then

$$P_p(T_0) > 0.$$

Proof. Let k be large enough s.t.

$$P(E_k) \geq \frac{1}{2}$$

where E_k is the event that at least 3 disjoint infinite clusters intersect B_k. We create a trifurcation in B_k as follows



- Let $\omega \in E_k$.
- Close all the edges with both extremities in B_k
- Consider x, y, z distinct, at distance k from 0, and connected to infinity (outside B_k)
- Consider a geodesic path γ_1 in B_k from x to y, and a geodesic path γ_2 in B_k from z to γ_1.
- Open all the edges in $\gamma_1 \cup \gamma_2$.

After this procedure, there exists a trifurcation in B_k. By insertion and deletion tolerance, we obtain (for some $c>0$)

$$c \cdot P_p(E_k) \leq P_p\Big(\bigcup_{u \in B_k} T_u\Big) \underset{\substack{\uparrow \\ \text{invariance}}}{\leq} |B_k| \; P(T_0) \qquad \blacksquare$$

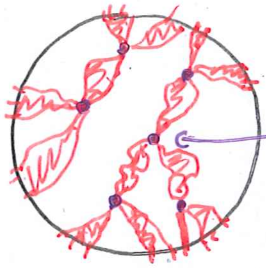
Proof of Thm:

Assume $P_p[N=\infty]=1$. Let (K_n) be connected sets s.t. $\frac{|\partial K_n|}{|K_n|} \xrightarrow[n\to\infty]{} 0$.

By Lemma 2, we have $c := P(T_0) > 0$.

Define $\mathcal{T}_n = \{ x \in K_n : x \text{ is a trifurcation} \}$.

By invariance $E[\mathcal{T}_n] = \sum_{x \in B_n} P(T_x) = c \cdot |K_n|$.



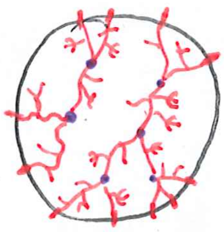
We claim that for every configuration $\mathcal{T}_n(\omega) \leq |\partial K_n|$.

To see this, consider the subgraph of K_n obtained by the following peeling procedure.

Let $F_0 = \{e_1, \dots, e_n\}$ open edges intersecting B_n.

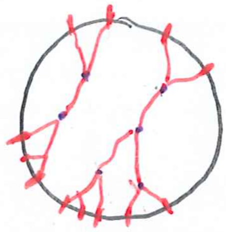
- For $i = 1, \dots, n$ set $F_i = \begin{cases} F_{i-1} \setminus \{e_i\} & \text{if } e_i \text{ belong to a cycle of } F_i \\ F_{i-1} & \text{otherwise.} \end{cases}$

After this first step the graph induced by F_n has no cycle and is therefore a forest.



Then remove all the edges of degree 1 in F_n (except the ones at the boundary of K_n). And repeat this operation until the time when there is no more edge of degree 1. ∎

Consider the graph induced by the remaining edges.



Write N_1 for the edges with degree 1 in this graph

$N_{\geq 3}$ ——— ≥ 3 ———

Notice that $N_1 \leq |\partial K_m|$ and $N_{\geq 3} \geq |\mathcal{T}_m|$ (because the trifurcation points have not been removed during the "peeling" procedure). By Applying Lemma 1 to each of the connected components of the graph above we obtain.

$$|\mathcal{T}_m| \leq N_{\geq 3} \leq N_1 \leq |\partial K_m|.$$

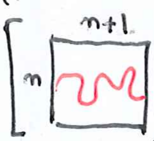
Taking the expectation, we obtain

$$c \cdot |K_m| \leq E\left[|\mathcal{T}_m|\right] \leq |\partial K_m|$$

which is a contradiction to $\frac{|\partial K_m|}{|K_m|} \xrightarrow[m\to\infty]{} 0$.

Corollary.

For $G = \mathbb{Z}^d$. If $\theta(p) > 0$, then

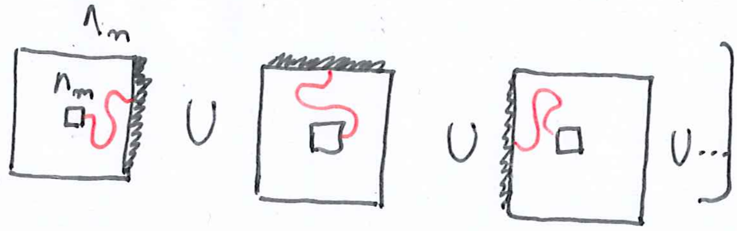
$\mathbb{P}_p\Big[$  $\Big] \xrightarrow[n\to\infty]{} 1.$

Not. $\Lambda_m = \{-m, \dots, m\}^d$

Proof (sketch):

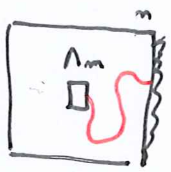
Fix $\varepsilon > 0$. Let $m \geq 1$ s.t.

$P_p\Big[$  $\Big] \geq 1-\varepsilon^{2d}$.

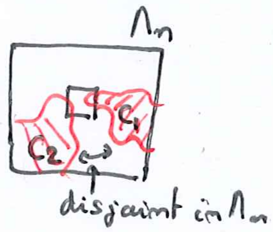
For $n \geq m$ $\quad P\Big[$  $\cup \dots \Big] \geq 1-\varepsilon^{2d}$

"Λ_m is connected inside Λ_n to one of the $2d$ facets of $\partial\Lambda_n$"

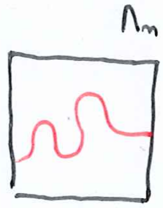
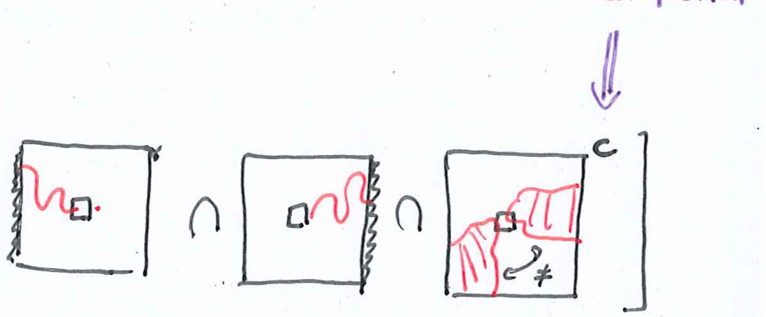
The square-root trick (see exercises) and rotation invariance imply

$P\Big[$  $\Big] \geq 1-\varepsilon$

Let n large enough s.t.

$P_p\Big[\exists C_1, C_2' :$  $\Big] \leq \varepsilon$

Conclusion:

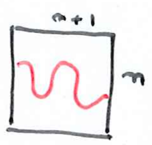
$P_p\Big[$  $\Big] \geq P_p\Big[$  $\Big]$

$\geq 1-3\varepsilon$ ∎

Corollary:

$$\boxed{p_c(\mathbb{Z}^2) = \frac{1}{2} \quad \text{and} \quad \theta(p_c) = 0.}$$

Proof: We have already proved $p_c \leq \frac{1}{2}$.

Since $P_{\frac{1}{2}}\Big($  $\Big) = \frac{1}{2} \underset{n\to\infty}{\not\longrightarrow} 1$, the corollary above gives $\theta\left(\frac{1}{2}\right) = 0$ hence $p_c \geq \frac{1}{2}$. ∎

3. Graphs with $p_c < p_u$.

Thm: (i) $p_c \leq \frac{1}{1+\phi}$. ($\phi$: isoperimetric constant)

(ii) $p_u \geq \frac{1}{\rho \cdot d}$. ($\rho$: spectral radius).

Proof (idea): Use exploration arguments:

(i) uses the law of large number on the variables used for the exploration and the isoperimetric inequality at each step of exploration.

(ii) uses that the exploration of C_0 is dominated by the trace of a branching random walk. ∎

Corollary: (i) If $\phi > \frac{d}{\sqrt{2}}$, then $p_c < p_u$.

(ii) If $\rho < \frac{1}{2}$, then $p_c < p_u$.

Proof: (i) Remind. $\rho^2 \leq 1 - \left(\frac{\phi}{d}\right)^2$

If $\left(\frac{\phi}{d}\right)^2 \geq \frac{1}{2}$ then $\rho^2 \leq \frac{1}{2} \leq \left(\frac{\phi}{d}\right)^2$

ie $\underline{d\rho \leq \phi}$.

Hence

$$p_c \leq \frac{1}{1+\phi} < \frac{1}{\phi} \leq \frac{1}{d\rho} \leq p_u$$

(ii) Remind. $\frac{\phi}{d} \geq 1-\rho$

If $\rho \leq \frac{1}{2}$, Then $\frac{\phi}{d} \geq \frac{1}{2} \geq \rho$ ie $\underline{d\rho \leq \phi}$

which implies $p_c < p_u$ (as above). ∎

Application:

For $k \geq 1$ define $G^{[k]}$ to be the graph with:

- vertex set V
- $\{x,y\}$ is an edge of $G^{[k]}$ if $\exists$ a walk of length <u>exactly k</u> from x to y.

Thm: If G is non-amenable then $\exists k \geq 1$ s.t.

$$p_c\left(G^{[k]}\right) < p_u\left(G^{[k]}\right).$$

Rk: $G^{[k]}$ is roughly-isometric to G.

Proof: Let $X_n^{[k]}$ SRW on $G^{[k]}$ and X_n S.R.W on G.

$\mathbb{P}_0\left[X_n^{[k]} = 0\right] = \mathbb{P}_0\left[X_{kn} = 0\right] = \rho^{kn+o(n)}$. Hence $\rho(G^{[k]}) = \underbrace{\rho(G)^k}_{(k \text{ large}) \to \leq \frac{1}{2}}$. ∎

CHAPTER 4:

– MASS TRANSPORT PRINCIPLE –

Ref: [LYONS-PERES, Chap. 8] [MARTINEAU, PhD-Thesis]

$G = (V, E)$ transitive, loc. finite, connected, infinite graph degree d, fixed origin $0 \in V$.

$\mathbb{P}_p = \prod_{e \in E} (p\,\delta_1 + (1-p)\,\delta_0)$ Bernoulli-percolation measure with density p.

1. MASS TRANSPORT PRINCIPLE.

1.1. UNIMODULAR TRANSITIVE GRAPHS.

Def: A function $f : V \times V \longrightarrow [0, \infty]$ is said to be <u>invariant</u> if

$$\forall \varphi \in \mathrm{Aut}(G) \quad f(\varphi \cdot x, \varphi \cdot y) = f(x, y)$$

"f is invariant under the diagonal action of $\mathrm{Aut}(G)$"

Def. We say that G is <u>unimodular</u> if for every $f : V \times V \to [0, \infty)$ invariant

$$\sum_{x \in V} f(0, x) = \sum_{x \in V} f(x, 0).$$

"mass transport principle"

Interpretation:
- $f(x, y)$ = "mass sent from x to y"
- $\sum_x f(0, x)$ = "mass sent from 0"
- $\sum_x f(x, 0)$ = "mass received by 0"

Rk: the mass transport principle is trivial if f is symmetric!

$\boxed{\text{Example: If } G = \mathrm{Cay}(\Gamma, S), \text{ then } G \text{ is unimodular.}}$

Proof: Write g an element of Γ and e the neutral of Γ.

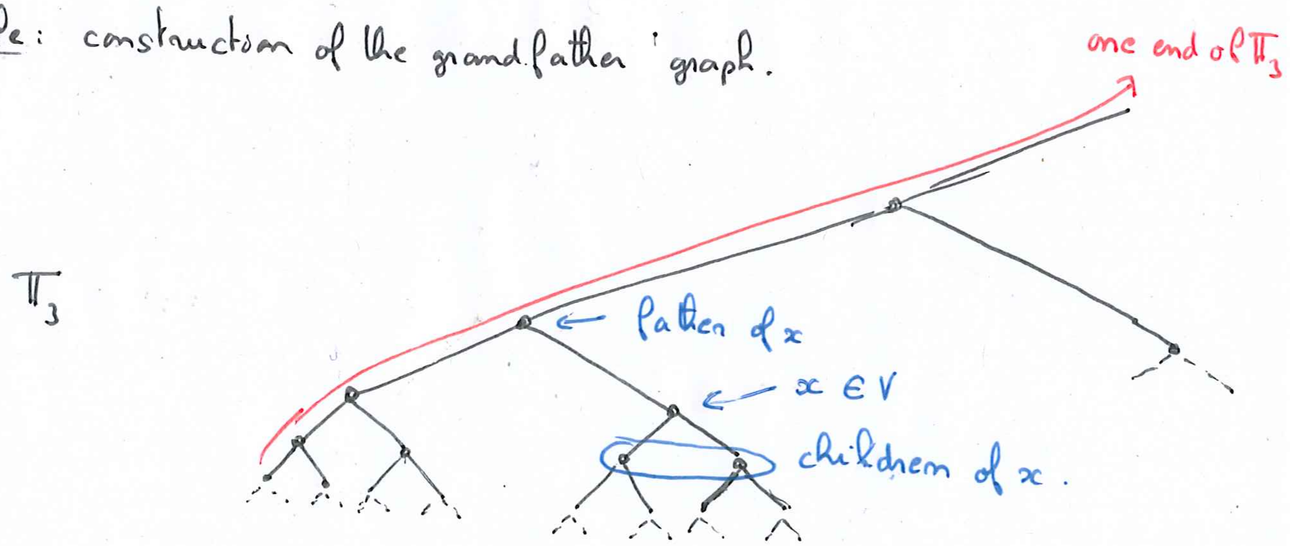
$$\sum_{x \in V} f(o,x) = \sum_{g \in \Gamma} f(e,g)$$

$$= \sum_{g \in \Gamma} f(g^{-1}, e) \qquad (f \text{ invariant})$$

$$= \sum_{g \in \Gamma} f(g,e)$$

$$= \sum_{x \in V} f(x,o) \quad \blacksquare$$

Example: construction of the grandfather graph.



By fixing one end of $\mathbb{T}_3$ one can see $\mathbb{T}_3$ as a "genealogical" tree in which each vertex has

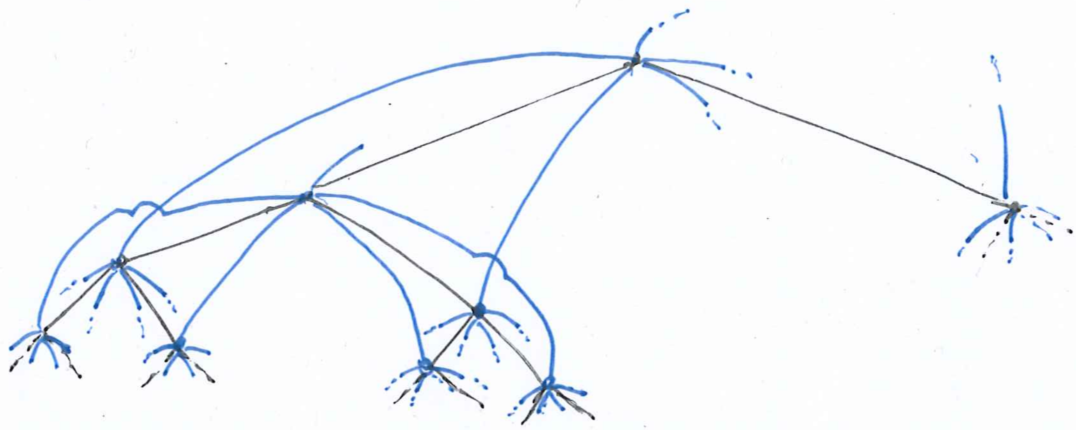
- 1 father
- 2 children.

Def: The <u>grand-father graph</u> is obtained from $\mathbb{T}_3$ by fixing one end of $\mathbb{T}_3$ and adding edges between

- the father of x
- the children of x

for every $x \in \mathbb{T}_3$.

"each vertex is connected to its grand father."



↳ the grand-father graph is a transitive graph of degree 8

Prop: $\boxed{\text{The grand-father graph is non-unimodular.}}$

Proof: Consider the mass transport $f(x,y) = \mathbb{1}_{x \text{ is the grand-father of } y}$

f is invariant (exercice) but

$$\underbrace{\sum_{x \in V} f(0,x)}_{=4} \neq \underbrace{\sum_{x \in V} f(x,0)}_{=1}.$$

Rk: In particular the grand-father graph is not a Cayley graph.

1.2 MASS-TRANSPORT FOR PERCOLATION.

To every $x, y \in V$ we associate a random variable $F_{x,y} : \{0,1\}^E \to [0, \infty]$.

We say that $F : V \times V \times \{0,1\}^E \to [0, \infty]$, $(x, y, \omega) \mapsto F_{x,y}(\omega)$ is invariant if

$$\forall \varphi \in \mathrm{Aut}(G) \quad F_{\varphi\cdot x, \varphi \cdot y}(\varphi\cdot\omega) = F_{x,y}(\omega).$$

Example. $F_{x,y}(\omega) = \mathbb{1}_{x \leftrightarrow y}(\omega)$ is invariant.

. $F_{x,y}(\omega) = \mathbb{1}_{x=0}$ is not invariant.

Prop. Assume G is unimodular.
If $\mathbb{P}$ invariant measure on $(\{0,1\}^E, \mathcal{F})$, and $F = (F_{x,y})$ invariant, then

$$\underbrace{\mathbb{E}\Big[\sum_{x \in V} F_{o,x}\Big]}_{\text{"expected mass sent"}} = \underbrace{\mathbb{E}\Big[\sum_{x\in V} F_{x,o}\Big]}_{\text{"expected mass received"}}$$

Proof: Let $f(x,y) = \mathbb{E}[F_{x,y}]$

For $\varphi \in \mathrm{Aut}(G)$
$$f(\varphi\cdot x, \varphi\cdot y) = \int_{\{0,1\}^E} F_{\varphi\cdot x, \varphi\cdot y}(\omega)\, dP(\omega)$$
$$\underset{P \text{ invariant}}{=} \int_{\{0,1\}^E} F_{\varphi\cdot x, \varphi\cdot y}(\varphi\cdot\omega)\, dP(\omega)$$
$$\underset{F \text{ invariant}}{=} \int_{\{0,1\}^E} F_{x,y}(\omega)\, dP(\omega) = f(x,y). \blacksquare$$

By applying the mass-transport principle to f and using Fubini theorem, we obtain the result. ∎

2 TWO APPLICATIONS.

Assume G is unimodular.

Assume $\mathbb{P}_p[N=\infty]=1$. We have seen that

$$c := \mathbb{P}_p(x \text{ trifurcation}) > 0.$$

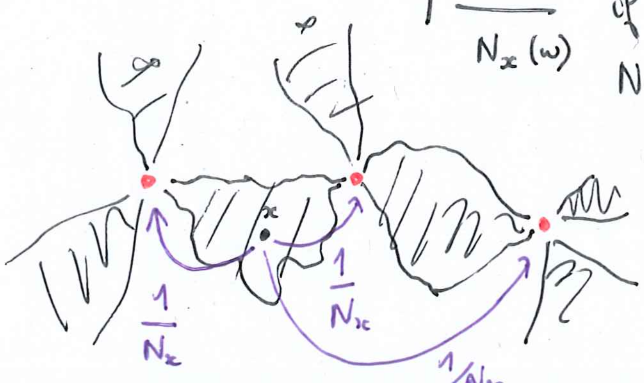
By ergodicity, there are infinitely many trifurcation in G a.s. (see exercices).

<u>Question</u>: What is the number of trifurcation in one infinite cluster?

<u>Prop</u>: If $\mathbb{P}$ invariant, then each cluster contains either 0 or infinitely many trifurcations.

<u>Proof</u>: Let $N_x(\omega)$ = number of trif. in $C_x(\omega)$.

Define
$$F_{x,y}(\omega) = \begin{cases} 0 & \text{if } N_x(\omega) \in \{0,\infty\}. \\ \dfrac{1}{N_x(\omega)} & \text{if } y \text{ trif of } C_x(\omega) \text{ and } N_x(\omega) \notin \{0,\infty\}. \end{cases}$$



F is invariant. Hence

$$\underbrace{\mathbb{E}\Big[\sum_{x\in V} F_{x,0}\Big]}_{= +\infty \times \mathbb{1}_{0 \text{ triv.}, N_0 \notin \{0,\infty\}}} = \underbrace{\mathbb{E}\Big[\sum_{x\in V} F_{0,x}\Big]}_{\leq 1}$$

i.e. $\mathbb{P}(0 \text{ triv}, N_0 \notin \{0,\infty\}) \times +\infty \leq 1$

Therefore $\forall x \quad \mathbb{P}[N_x \notin \{0,\infty\}] = 0$

<u>Prop</u>: Let $\mathbb{P}$ invariant measure on $\{0,1\}^E$.
Assume $\min_{e\in E} \mathbb{P}[e \text{ is open}] > 1 - \frac{\phi}{d}$, then

$$\mathbb{P}[0 \longleftrightarrow \infty] > 0$$

<u>Rk</u>: • Implies $p_c \leq 1 - \frac{\phi}{d}$.

• No hypothesis on $\mathbb{P}$ except invariance!!

<u>Proof</u>: Assume $\mathbb{P}[0 \longleftrightarrow \infty] = 0$. We prove $\mathbb{P}[e \text{ is open}] \leq 1 - \frac{\phi}{d}$.

Define $\deg_\omega(x) = \sum_{y\sim x} \mathbb{1}_{\omega(xy)=1}$.

For every $x, y \in V$, define

$$F_{x,y}(\omega) = \begin{cases} 0 & \text{if } |C_x(\omega)| = +\infty, \\ \frac{1}{|C_x(\omega)|} \mathbb{1}_{C_x(\omega) = C_y(\omega)} \cdot \deg_\omega(x) & \text{if } |C_x(\omega)| < \infty \end{cases}$$

If $C_0(\omega)$ is finite, we have

- $\sum_{x \in V} F_{0,x}(\omega) = \deg_\omega(x)$
- $\sum_{x \in V} F_{x,0}(\omega) = \frac{1}{|C_0(\omega)|} \sum_{x \in C_0(\omega)} \deg_x(\omega)$

$$= \frac{1}{|C_0(\omega)|} \left(d \cdot |C_0(\omega)| - |\partial C_0(\omega)| \right)$$

$$= d - \frac{|\partial C_0(\omega)|}{|C_0(\omega)|} \leqslant d - \phi$$

Since $C_0(\omega)$ is finite a.s., M.T.P. implies

$$\underbrace{\mathbb{E}[\deg_\omega(x)]}_{\geqslant\, d \times \min_e \mathbb{P}[e \text{ is open}]} \leqslant d - \phi$$

which gives $\min_e \mathbb{P}[e \text{ is open}] \leqslant 1 - \frac{\phi}{d}$ ∎

3. INDISTINGUISHABILITY.

Assume that G is unimodular.

<u>Idea</u>: If a percolation process produces infinitely many infinite clusters, is it possible that two infinite clusters "look" differently?

<u>Def</u>: A <u>cluster property</u> is a measurable function

$$\mathcal{P} : V \times \{0,1\}^E \longrightarrow \{\text{true}, \text{false}\}$$
$$(x, \omega) \longmapsto \mathcal{P}_x(\omega)$$

s.t. • $\forall \varphi \in \mathrm{Aut}(G)$ $\quad \mathcal{P}_{\varphi \cdot x}(\varphi \cdot \omega) = \mathcal{P}_x(\omega)$ ("invariance")

• If $C_x(\omega) = C_y(\omega)$ then $\mathcal{P}_x(\omega) = \mathcal{P}_y(\omega)$ ("$\mathcal{P}$ constant on clusters")

Example: . $\mathcal{P}_x(\omega)$ = "the cluster of x has infinitely many trifurcations."

Non-examples: . $\mathcal{P}_x(\omega)$ = "all the edges adjacent to x are open in ω."

$\rightarrow$ not constant on clusters

. $\mathcal{P}_x(\omega)$ = "$0 \longleftrightarrow x$ in ω" $\rightarrow$ not invariant.

Thm (Indistinguishability of infinite clusters):

Assume that G is unimodular.
Let $\mathbb{P}$ be an invariant, insertion-tolerant measure on $\{0,1\}^E$.
Then, for every cluster property $\mathcal{P}$, we have

$$\mathbb{P}\Big[(\forall x \in V_\infty \ \mathcal{P}_x(\omega) = \text{true}) \cup (\forall x \in V_\infty \ \mathcal{P}_x(\omega) = \text{false})\Big] = 1$$

where $V_\infty = \{x \in V : x \longleftrightarrow \infty \text{ in } \omega\}$.

Proof: admitted, see [Lyons-Schramm, 1999]

Application 1

Assume $\mathbb{P}_p(N = \infty) = 1$.
Then every infinite cluster contains infinitely many trifurcations.

Proof:

We already know that, a.s.,

- there exist infinitely many trifurcation in G.
- each infinite cluster contains either 0 or infinitely many trifurcation.

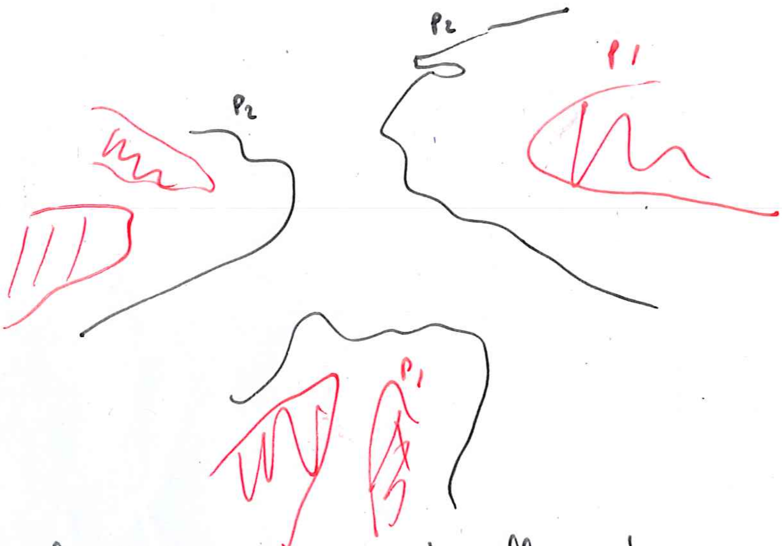
By indistinguishability, each infinite cluster contains infinitely many trifurcation. ∎

Appli 2 :

Let $p_1 < p_2$.

(i) If $\mathbb{P}_{p_1}[N=1]=1$, then $\mathbb{P}_{p_2}[N=1]=1$

(ii) If $\mathbb{P}_{p_1}[N\geq 1]=1$, then in the standard coupling*, every infinite p_2-cluster contains an infinite p_1-cluster.



Proof: Since (ii) $\Rightarrow$ (i), it suffices to prove (ii)

Consider the cluster property

$\mathcal{J}_x(\omega)$ = " $\frac{p_1}{p_2}$ - percolation on $C_x(\omega)$ produce an infinite cluster as".

If $\mathbb{P}_{p_2}\left(\forall x \in V_\infty \quad \mathcal{J}_x(\omega) = \text{false}\right)$ then $\mathbb{P}_{p_1}(N\geq 1)=0$

(Indeed a p_1-percolation on G can be obtained by doing a $\frac{p_1}{p_2}$-percolation on (V,ω) where ω is a p_2-percolation on G. If $\forall x \ \ \mathcal{J}_x(\omega) = \text{false}$, then the resulting configuration has only finite components.)

* Let $(U_e)_{e\in E}$ iid uniform on $[0,1]$. Call p-cluster a connected component of the graph induced by $\omega = (\mathbb{1}_{U_e\leq p})_{e\in E}$.

Therefore, by indistinguishability, we must have

$$\mathbb{P}_{p_2}\left[\forall x \in V_\infty \quad \mathcal{O}_x(\omega) = \text{true}\right].$$

The result follows. ∎

Application 3. A criterion for non-uniqueness

Thm: Assume $\mathbb{P}_p[N \geq 1] = 1$. Then

$$\mathbb{P}_p[N = \infty] = 1 \iff \inf_{x \in V} \mathbb{P}_p[0 \longleftrightarrow x] = 0$$

Proof (sketch) for a detailed proof, see LYONS-SCHRAMM, Thm 4.1

$\boxed{\Leftarrow}$ Assume $\mathbb{P}_p[N=1] = 1$. By invariance, we have

$$\forall x \in V \quad \mathbb{P}_p[x \longleftrightarrow \infty] = \theta > 0.$$

The following computation concludes the proof

$$\mathbb{P}_p[0 \longleftrightarrow x] \geq \mathbb{P}_p[0 \longleftrightarrow \infty, x \longleftrightarrow \infty, N = 1]$$

$$\overset{(N=1\text{ a.s.})}{=} \mathbb{P}_p[0 \longleftrightarrow \infty, x \longleftrightarrow \infty]$$

$$\overset{(FKG)}{\geq} \theta^2.$$

$\boxed{\Rightarrow}$ Assume $\inf_{x \in V} \mathbb{P}_p[0 \longleftrightarrow x] = \tau > 0$

Let (X_n) be a SRW from 0, independent of the percolation configuration.

We claim that $\exists \beta \in [0,1]$ s.t. a.s. $\forall C$ infinite cluster

$$\underbrace{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{X_k \in C}}_{\simeq \text{ "density of } C\text{"}} = \beta . \quad (*)$$

This is proved in 2 steps:

1. Using ergodicity of P_p and a subadditive lemma, it is possible to prove that a.s. $\forall C$ infinite cluster.

$$\alpha(C) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{X_k \in C} \quad \text{exists.}$$

2. Using the cluster property

$$\mathcal{J}_x^\beta(\omega) = \{ \alpha(C_x(\omega)) \leq \beta \}$$

and indistinguishability, we find that $\exists \beta \in [0,1]$ s.t.

$$\forall x \in V_\infty \quad \alpha(C_x(\omega)) = \beta \quad \text{a.s.}$$

Now, by Dominated convergence

$$\underbrace{\mathbb{E}[\alpha(C_0)]}_{= \theta \times \beta} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \underbrace{\mathbb{P}[X_k \in C_0]}_{\geq \tau} \geq \tau > 0$$

we obtain that a.s. for every C infinite cluster $\alpha(C) = \beta \geq \frac{\tau}{\theta} > 0$

Since $\sum_{C \text{ infinite cluster}} \alpha(C) \leq 1$ a.s. we must have $N = 1$ a.s. ∎

Rk: This criterion provides us with an alternative proof of

$$\mathbb{P}_{p_1}[N=1]=1 \Rightarrow \mathbb{P}_{p_2}[N=1]=1 \quad \text{when } p_1 \leq p_2 .$$

Indeed

$$\mathbb{P}_{p_1}[N=1]=1 \Rightarrow \begin{cases} \mathbb{P}_{p_1}[N\geq 1]=1 \\ \inf_x \mathbb{P}_{p_1}[0\leftrightarrow x]>0 \end{cases} \Rightarrow \begin{cases} \mathbb{P}_{p_2}[N\geq 1]=1 \\ \inf_x \mathbb{P}_{p_2}[0\leftrightarrow x]>0 \end{cases} \Rightarrow \mathbb{P}_{p_2}[N=1]=1$$

Rk: We find that

$$p_u = \inf\{p : \inf_x \mathbb{P}_p[0\leftrightarrow x]>0\} .$$

Corollary: For every $d\geq 3$

$$\boxed{p_u(\mathbb{T}^d \times \mathbb{Z}) \leq \frac{1}{2} .}$$

Proof: Let x be a vertex of $\mathbb{T}^d\times\mathbb{Z}$. One can find a subgraph H of $\mathbb{T}^d$ that is isomorphic to $\mathbb{Z}^2$ and contains both o and x. (consider $H = R\times\mathbb{Z}$, where $R\times\{0\}$ is an infinite path on $\mathbb{T}^d\times\{0\}$ containing o and the projection of x on $\mathbb{T}^d\times\{0\}$)

Hence for every $p>\frac{1}{2}$ $\mathbb{P}_p[o\leftrightarrow x]\geq \theta^{\mathbb{Z}^2}(p)^2 >0$ ■