READING GROUP 2017 PART I: ZEROS OF GAUSSIAN ANALYTIC FUNCTIONS

In the first part of the reading group we will be studying the zero sets of Gaussian analytic functions (GAF).

Definition 1. *Let Λ be a complex domain (for us it will be either $\mathbb{D}$ or $\mathbb{C}$). Then $F(z) : \Lambda \to \mathbb{C}$ is a Gaussian analytic function (GAF) on Λ if it is almost surely analytic on Λ and if for any $z_1, \dots, z_n$ the vector $(F(z_1), \dots, F(z_n))$ has a Gaussian distribution.*

It is easy to see that such functions exist: one can just take $(\xi_n)_{n\in\mathbb{N}}$ to be i.i.d standard normals and set

$$F(z) = \sum_{n\geq 0} \xi_n c_n z^n.$$

Choosing sufficiently rapid decay of c_n then guarantees any wished radius of convergence. For example:

- $F_{\mathbb{D}}(z) = \sum_{n\geq 0} \xi_n z^n$ is analytic on D - we call this the hyperbolic GAF.
- $F_{\mathbb{C}}(z) = \sum_{n\geq 0} \xi_n \frac{1}{\sqrt{n!}} z^n$ is an entire function - we called it the planar GAF.

We will study the zero sets $\mathcal{Z}_F$ of the GAF F and basically ask how it looks like, and how it compares to other known point processes like for example the Poisson point process.

Isometry invariant zero sets. A first cute observation is that the zero sets $\mathcal{Z}_{F_{\mathbb{D}}}$ and $\mathcal{Z}_{F_{\mathbb{C}}}$ are invariant in law with respect to Mobius transformations of the disk and planar isometries respectively. A deeper result says that they are essentially the unique Gaussian analytic functions with this property:

Theorem 2 (Rigidity of the GAF). *$F_{\mathbb{D}}(z)$ is (essentially) the only GAF on $\mathbb{D}$ whose zero set is invariant in law w.r.t. Mobius transformations of the disk. Similarly, $F_{\mathbb{C}}(z)$ is (essentially) the only GAF whose zero set is invariant in law w.r.t. isometries of the plane.*

Here essentially means that there is one free parameter, and there is also the possibility to multiply by e^g for any fixed analytic function g. Proving this theorem and some basic properties about the zero set of the planar GAF is the content of the first two lectures. We will follow the book "Zeros of Gaussian Analytic Functions and Determinantal Point Processes" by Hough, Krishnapur, Peres and Virag.

Zero set of the hyperbolic GAF. Thereafter we will look closer at the zero set of the hyperbolic GAF. It comes out that this is the only model whose zero set has a determinantal structure. This means that given disjoint sets $D_i \subset \mathbb{D}$ and denoting by $\mathcal{Z}_F(\mathbb{D})$ the number of points in a set $\mathbb{D}$, we can calculate:

$$\mathbb{E}\left[\mathcal{Z}_{F_{\mathbb{D}}}(D_1) \times \dots \times \mathcal{Z}_{F_{\mathbb{D}}}(D_n)\right] = \pi^{-n} \int_{D_1\times\dots\times D_n} \det\left[\frac{1}{(1-z_i\overline{z_j})^2}\right]_{i,j} dz_1 \dots dz_n,$$

where by dz_i we denote the Lebesgue measure. Sometimes one calls $\rho(z_1, \dots, z_n) = \det\left[\frac{1}{(1-z_i\overline{z_j})^2}\right]_{i,j}$ the joint intensities / correlation functions $\rho(z_1, \dots, z_n)$.

This determinantal structure, that is not present in the planar case allows to find precise results. For example it follows that

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- $\{|z| : F_{\mathbb{D}}(z) = 0\}$ has the same law as $\{U_k^{1/2k}\}$ where U_i are i.i.d $U[0,1]$;
- the "hole probability", i.e. that there are no points in the disk of radius r is equal to $\exp(\frac{-\pi^2+o(1)}{12(1-r)})$ as $r \to 1$.

Again, we basically follow the above-mentioned book.

Hole probability for the planar GAF. Our next task might be to find the hole probability for the planar GAF. More precisely to show that

Theorem 3 (Hole probability for the planar GAF). *The probability p_R that there the planar GAF has no zeros in the disk of radius R satisfies $\exp(-cR^4) \leq p_R \leq \exp(-CR^4)$.*

This should be compared with the same probability for the PPP, in which case the exponent is a constant times R^2. A toy model that explain this behaviour is the following: take $\xi_{i,j}$ to be i.i.d standard normal random variables and look at the point process $T := \{(i,j) + \xi_{i,j}\}$. It is easy to see that the probability q_r that T has no points in the disk of radius r also satisfies $\exp(-c_1 R^4) \leq q_R \leq \exp(-c_2 R^4)$.

Allocation / matching / transport. This toy model, although one can see it doesn't encompass for example the right level of fluctuations of the zero set, hints in the right direction. Indeed, the zero set of the planar GAF can be seen as a perturbed lattice, the perturbations are only not exactly Gaussian and certainly not independent:

Theorem 4 (Zeros of the planar GAF as a perturbed lattice). *There exists random variables $\xi_{i,j}$ with $i, j \in \mathbb{Z}$ such that*

- *$\{(i,j) + \xi_{i,j}\}$ has the same law as $\mathcal{Z}_{F_{\mathbb{C}}}$;*
- *$\xi_{i,j}$ is invariant under shifts of $\mathbb{Z}^2$;*
- *for some positive c we have $\mathbb{E}e^{c\xi_{0,0}^2} < \infty$.*

This result can also be seen as a matching result: we are matching the zero set of the planar GAF to the lattice points. It could also be seen as a transport result, transporting a unit mas s from each zero to each lattice point. A similar task would be to transport the mass of the zero set (seen as a sum of Dirac masses) to the Lebesgue measure. In fact there is an explicit way to do this: consider the potential $u(z) = \log|F_{\mathbb{C}}(z)| - \frac{1}{2}|z|^2$ reminding the 2D gravitational potential. One can observe that the local minima of the potential u correspond to the zeros of $F_{\mathbb{C}}$. Thus when one considers the ODE

$$\frac{dZ(t)}{dt} = -\nabla u(Z(t)),$$

then in fact any point will flow to one of the zeros. The final result we would like to discuss describes the basins of attraction for all the individual zeros:

Theorem 5 (Gravitational allocation). *For the ODE described above:*

- *the boundaries of the basins of attractions are finite unions of smooth curves;*
- *the basins of attraction divide $\mathbb{C}$ into connected cells of equal area π;*
- *for any point $z \in \mathbb{C}$, the probability that the diameter of the basin of attraction containing this point is larger than R can be bounded by $c\exp(-CR(\log R)^{3/2})$ and $C\exp(-cR\sqrt{\log R})$.*

Time permitting, we will also try to see this result from the point of view of optimal transport of measures.