Definition of Gaussian analytic functions (GAFs)

A Gaussian analytic function (GAF) will be defined as a random element of the space of analytic functions on a domain in the complex plane following a certain distribution.

For the definition we need the complex Gaussian distribution, so let's start by recalling it.

in words

Definition 1: • $\xi \sim N_{\mathbb{C}}^1(0,1)$ standard complex Gaussian if ξ is a complex-valued random variable with density $\frac{1}{\pi} e^{-|z|^2}$ w.r.t. the Lebesgue measure.
(alternatively $\xi = X + iY$ with $X, Y \sim N_{\mathbb{R}}(0, \frac{1}{2})$ independent)

• $\underline{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \sim N_{\mathbb{C}}^n(0, I)$ standard complex Gaussian vector if $\xi_1, \dots, \xi_n$ i.i.d. $\sim N_{\mathbb{C}}^1(0,1)$

• $\underline{\zeta} = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_m \end{pmatrix} \sim N_{\mathbb{C}}^m(\mu, \Sigma)$ ~~standard~~ complex Gaussian vector with mean μ and covariance Σ if $\underline{\zeta} = B\underline{\xi} + \mu$ with $\underline{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$, B $m \times n$ matrix, μ vector of length m and $\Sigma = BB^*$.

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Remark 1: • As in the real case, the mean vector and the covariance matrix of a complex Gaussian vector determine its distribution

• If $\underline{\xi} \sim N_{\mathbb{C}}^{n}(\mu, \Sigma)$, then $\forall_{j,k=1,\dots n}$ $\mathbb{E}[(\xi_j-\mu_j)\overline{(\xi_k-\mu_k)}] = \Sigma_{jk}$
$\mathbb{E}[(\xi_j-\mu_j)(\xi_k-\mu_k)] = 0$

maybe skip this

• If $\xi \sim N_{\mathbb{C}}^{1}(0,1)$, then $|\xi|^2 \sim Exp(1)$

$$\left(\mathbb{P}[|\xi|^2 \leq x] = \int_{\{z\in\mathbb{C} \mid |z|^2\leq x\}} \frac{1}{\pi} e^{-|z|^2} dz = \int_0^{2\pi}\int_0^{\sqrt{x}} \frac{1}{\pi} e^{-r^2} r\,dr\,d\varphi = \dots = 1-e^{-x} \right)$$

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Now consider the measurable space given by $\{f: \Lambda \to \mathbb{C} \text{ analytic}\}$ where $\Lambda \subseteq \mathbb{C}$ endowed by the Borel σ-algebra $\mathcal{B}$ formed by the topology of uniform convergence on compact subsets of Λ, i.e. the σ-algebra generated by the open sets in this topology.

Then $(\{f: \Lambda \to \mathbb{C} \text{ analytic}\}, \mathcal{B})$ is complete, separable, metric space

induced by the seminorms $\sup_{z\in K}|f(z)|$ for $K \subseteq \Lambda$ compact

if $(f_n)_n$ Cauchy sequence, then $f_n \to f$ with f continuous since uniformly convergent on compact subsets and moreover for any $\gamma \subseteq \Lambda$ closed curve one has

$$\int_\gamma f\,dz = \int_\gamma \lim f_n\,dz \underset{\substack{\text{uniform}\\ \text{on compact}}}{=} \lim \int_\gamma f_n\,dz \underset{f_n \text{ analytic}}{=} 0$$

$\Rightarrow f$ analytic by Morera's theorem

Definition 2: Let F be a random variable (in some abstract probability space) taking values in $\{f : \Lambda \to \mathbb{C} \text{ analytic}\}$. Then F is called GAF on Λ if $(F(z_1), \dots, F(z_n)) \sim \mathcal{N}_{\mathbb{C}}^n(0, \Sigma)$ for some Σ for any $z_1, \dots, z_n \in \Lambda$, $n \geq 1$

Remark 2: If F GAF on Λ, then for any $z_1, \dots z_n \in \Lambda$ on has that $(F(z_1), \dots F(z_n))$ is determined by some Σ where $\Sigma_{jk} := K(z_j, z_k)$, see Remark 1.

$\Rightarrow$ Function $K(\cdot, \cdot)$ determines all finite dimensional marginals of F

$\underset{F \text{ a.s. continuous}}{\Longrightarrow}$ $K(\cdot, \cdot)$ on $\Lambda \times \Lambda$ determines distribution of F

Two Examples

Examples: Let $L > 0$ and ξ_n i.i.d. $\sim \mathcal{N}_{\mathbb{C}}^1(0, 1)$.

1) $F_{\mathbb{C}}(z) := \sum_{k=0}^{\infty} \xi_k \sqrt{\frac{L^k}{k!}} z^k$ is a GAF on $\mathbb{C}$ with covariance function $K_{F_{\mathbb{C}}}(z, w) = e^{Lz\bar{w}}$ $\forall z, w \in \mathbb{C}$

2) $F_{\mathbb{D}}(z) := \sum_{k=0}^{\infty} \xi_k \sqrt{\frac{L(L+1)\dots(L+k-1)}{k!}} z^k$ is a GAF on $\mathbb{D} = \{|z| < 1\}$ unit disc with covariance function $K_{F_{\mathbb{D}}}(z, w) = (1 - z\bar{w})^{-L}$ $\forall z, w \in \mathbb{D}$

Indeed

- $F_{\mathbb{C}}, F_{\mathbb{D}}$ are analytic within their radius of convergence. We have by Cauchy-Hadamard

$$r_1 = \frac{1}{\limsup_{n\to\infty} \left|\xi_n \sqrt{\frac{L^n}{n!}}\right|^{\frac{1}{n}}} = \frac{1}{\sqrt{L}\ \limsup_{n\to\infty} \underbrace{|\xi_n|^{\frac{1}{n}}}_{\xrightarrow{a.s.} 1} \underbrace{\left(\frac{1}{n!}\right)^{\frac{1}{2n}}}_{\to 0 \text{ by Stirling}}} = \infty \quad \text{a.s.}$$

$$r_2 = \frac{1}{\limsup_{n\to\infty} \left|\xi_n \sqrt{\frac{L(L+1)\dots(L+n-1)}{n!}}\right|^{\frac{1}{n}}} = \frac{1}{\limsup_{n\to\infty} \underbrace{|\xi_n|^{\frac{1}{n}}}_{\xrightarrow{a.s.} 1} \underbrace{\left(\frac{L(L+1)\dots(L+n-1)}{n!}\right)^{\frac{1}{2n}}}_{\longrightarrow 1}} = 1 \quad \text{a.s.}$$

where we used $\underbrace{|\xi_n|}_{\sim \mathcal{N}_{\mathbb{C}}(0,1)}^{\frac{1}{n}} \longrightarrow 1$ a.s.

maybe skip this and say it can be shown by Borel-Cantelli argument

$$\Big(\mathbb{P}\left[\lim_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1\right] = \mathbb{P}\Big[\lim_{n\to\infty} \overbrace{\frac{1}{2n} \log(\underbrace{|\xi_n|^2}_{\sim \mathrm{Exp}(1) \text{ by Remark 1}})}^{=: X_n} = 0\Big] = 1$$

since $\mathbb{E}[X_n] \longrightarrow 0$

$$\sum_{n=0}^{\infty} \mathrm{Var}(X_n) < \infty$$

$$\Rightarrow \forall \varepsilon > 0, \quad \sum_{n=0}^{\infty} \mathbb{P}[|X_n| > \varepsilon] \le \sum_{n=0}^{\infty} \overbrace{\mathbb{P}[|X_n - \mathbb{E}[X_n]| > \varepsilon]}^{\le \frac{1}{\varepsilon^2}\mathrm{Var}(X_n)} + \sum_{n=0}^{\infty} \underbrace{\mathbb{P}[|\mathbb{E}[X_n]| > \varepsilon]}_{\substack{\in \{0,1\} \text{ and } = 0 \\ \text{for } n \text{ large enough} \\ \text{since } \mathbb{E}[X_n] \to 0}} < \infty$$

$$\overset{\text{Borell-Cantelli}}{\Longrightarrow} \mathbb{P}[|X_n| > \varepsilon \text{ happens infinitely many times}] = 0$$

$$\Rightarrow \underbrace{\mathbb{P}[|X_n| \le \varepsilon \ \forall n \text{ large enough}] = 1}$$

$$\xrightarrow[\text{monotonicity}]{\varepsilon \to 0} \mathbb{P}[\forall \varepsilon > 0, \ |X_n| \le \varepsilon \ \forall n \text{ large enough}] \Big)$$

Note: Inside the radius of convergence they converge uniformly on compact subsets

- For $z_1, \dots z_n \in \mathbb{C}/\mathbb{D}$ and $a_1, \dots a_n \in \mathbb{C}$

$a_1 F_{\mathbb{C}/\mathbb{D}}(z_1) + \dots + a_n F_{\mathbb{C}/\mathbb{D}}(z_n) \sim$ centered complex Gaussian (since limits of Gaussians are Gaussians)

$\Rightarrow F_{\mathbb{C}} / F_{\mathbb{D}}$ are GAF on $\mathbb{C}/\mathbb{D}$

- Covariance function

$$K_{F_{\mathbb{C}}}(z,w) = \mathbb{E}[F_{\mathbb{C}}(z)\overline{F_{\mathbb{C}}(w)}] = \sum_{n=0}^{\infty} \frac{L^n}{n!}(z\bar{w})^n = e^{Lz\bar{w}}$$

$\mathbb{E}[\varepsilon_n \bar{\varepsilon}_m] = \delta_{nm}$

Binomial series (wiki)

$$K_{F_{\mathbb{D}}}(z,w) = \mathbb{E}[F_{\mathbb{D}}(z)\overline{F_{\mathbb{D}}(w)}] = \sum_{n=0}^{\infty} \underbrace{\frac{L(L+1)\dots(L+n-1)}{n!}}_{=: \text{ generalized binomial coefficient } \binom{L+n-1}{n}} (z\bar{w})^n = (1-z\bar{w})^{-L}$$

(Since $K_{F_{\mathbb{D}}}(z,z) = \mathbb{E}[|F_{\mathbb{D}}(z)|^2] \geq 0$, one has for $L \notin \mathbb{N}$ to define $(1-z\bar{w})^{-L} \underset{\text{def}}{=} \exp(-L(\ln|1-z\bar{w}| + i\arg(1-z\bar{w})))$ via the branch with $\arg(1-|z|^2)=0$)

These two GAFs will be the two main examples for later. How do other GAFs look like? Or say, how can one construct GAFs? The following Lemma, of which I will only sketch the proof, shows that a certain class of (random) power series with i.i.d. coefficients are GAFs, including the above two examples.

in words

Lemma: Let $(\Psi_k)_k$ analytic functions on Λ (deterministic) such that $\sum_{k=1}^{\infty} |\Psi_k(\cdot)|^2$ converges uniformly on compact subsets of Λ. Let $(\xi_k)_k$ i.i.d. $\sim \mathcal{N}_{\mathbb{C}}^1(0,1)$.

$\Rightarrow \sum_{k=1}^{\infty} \xi_k \Psi_k(\cdot)$ is a GAF on Λ with covariance function $K(z,w) = \sum_{k=1}^{\infty} \Psi_k(z) \overline{\Psi_k(w)}$.

Proof: (Sketch)

1st step: Let $K \subseteq \Lambda$ compact and $X_n := \sum_{k=1}^{n} \xi_k \Psi_k \in L^2(K)$.
Show $\mathbb{E}[\|X_n\|^2_{L^2(K)} \mid \xi_1, \dots \xi_m] = \|X_m\|^2_{L^2(K)} + \sum_{k=m+1}^{n} \|\Psi_k\|^2_{L^2(K)}$ $\forall m \leq n$

2nd step: Set $\tau := \inf\{n \geq 1 \mid \|X_n\|_{L^2(K)} \geq \varepsilon\}$ stopping time.
Show $\mathbb{E}[\|X_n\|^2_{L^2(K)}] \geq \varepsilon^2 \mathbb{P}[\tau \leq n]$ by conditioning on τ and using 1st step

$\Rightarrow \mathbb{P}[\sup_{k \leq n} \|X_n\| \geq \varepsilon] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \|\Psi_k\|^2_{L^2(K)}$

$\xRightarrow{\text{apply this to sequence}} \dots \Rightarrow$ $(X_{N+k} - X_N)_k$ $\mathbb{P}[\sup_{n,m \geq N} \|X_m - X_n\|_{L^2(K)} \geq \varepsilon] \xrightarrow{N \to \infty} 0$ $\forall \varepsilon > 0$

using assumption on summability of $|\Psi_k(\cdot)|^2$ uniformly on compacts

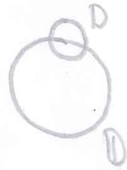
$\Rightarrow$ a.s., $(X_n)_n$ is Cauchy sequence in $L^2(K)$, hence $X_n \longrightarrow X$ in $L^2(K)$

3rd step: Use Cauchy's formula $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(u)}{u-z} du$ for the analytic functions X_n and $X_n \to X$ in $L^2(K)$ together with some averaging to show $X_n \longrightarrow X$ uniformly on compact subsets of Λ

skip this

Remark 3: One can show that the unit disk is truly the domain of $F_{\mathbb{D}}$ in the sense that it doesn't have an analytic continuation to a larger domain. This is easy to see.

Let $D \subseteq \mathbb{C}$ open disk, $D \not\subseteq \mathbb{D}$, $D \cap \mathbb{D} \neq \emptyset$.



$\Rightarrow \mathbb{P}[\, F_{\mathbb{D}}$ has analytic continuation to $D \cup \mathbb{D}\,] \in \{0,1\}$

it is a tail event
does not depend on $\xi_1, \dots \xi_n$
for any finite n and ξ_i indep.
$\Rightarrow$ use Kolmogorov 0-1 law

Assume by contradiction it is $= 1$.

$\Rightarrow \mathbb{P}[\, F_{\mathbb{D}}$ has analytic continuation to $D' \cup \mathbb{D}\,] = 1$
for any rotation D' of D around 0

$\mathcal{N}^1_{\mathbb{C}}(0,1)$ is rotationally symmetric

$\Rightarrow \mathbb{P}[\, F_{\mathbb{D}}$ has analytic continuation to  $] = 1$

choose finitely many rotations

$\Rightarrow \mathbb{P}[$ radius of convergence of $F_{\mathbb{D}}$ is $> 1\,] = 1$ ↯

do this in words and with drawings

Invariance of zero sets of $F_{\mathbb{C}}$, $F_{\mathbb{D}}$ under isometries

In the rest of this presentation we will see that the distribution of the zero set of the above defined $F_{\mathbb{C}}$ and $F_{\mathbb{D}}$ is invariant under certain transformations. And then we show that actually these $F_{\mathbb{C}}$ and $F_{\mathbb{D}}$ are essentially the only GAFs on their respective domains which have this invariance in distribution property of their zero set.

in words

Recall:

- The isometries of $\mathbb{C}$ under the Euclidean metric are given by $z \longmapsto \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$ (translations, rotations and combinations of them)

these isometries also preserve the Lebesgue measure of subsets of $\mathbb{C}$

- The isometries of $\mathbb{D}$ under the hyperbolic metric ("Poincaré disk") are given by $z \longmapsto \alpha \frac{z+\beta}{\bar{\beta} z + 1}$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$, $|\beta| < 1$

(the metric is $\delta(z,z') := \frac{2|z-z'|^2}{(1-|z|^2)(1-|z'|^2)}$ $\forall z, z' \in \mathbb{D}$)

these isometries also preserve the hyperbolic area measure of subsets of $\mathbb{D}$

Proposition: (i) The zero set of $F_{\mathbb{C}}$ is invariant in distribution under euclidean isometries of $\mathbb{C}$

(ii) The zero set of $F_{\mathbb{D}}$ is invariant in distribution under hyperbolic isometries of $\mathbb{D}$.

Proof: Let's do the proof only for $F_{\mathbb{C}}$, it is nicer to write down but same in spirit as for $F_{\mathbb{D}}$.

So let $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$ and $\varphi(z) = \alpha z + \beta$.

Set $G = F_{\mathbb{C}} \circ \varphi$, is also a GAF on $\mathbb{C}$.

We need to show $\{z \in \mathbb{C} \mid F_{\mathbb{C}}(z) = 0\} \overset{d}{=} \{z \in \mathbb{C} \mid G(z) = 0\}$.

The covariance function of G is

$$K_G(z,w) = \mathbb{E}\left[F_{\mathbb{C}}(\alpha z + \beta) \overline{F_{\mathbb{C}}(\alpha w + \beta)} \right]$$

ξ_k iid $\sim \mathcal{N}_{\mathbb{C}}(0,1)$

$$= \sum_{k=0}^{\infty} \frac{L^k}{k!} \underbrace{(\alpha z + \beta)^k \overline{(\alpha w + \beta)^k}}_{= (z\bar{w} + \alpha\bar{\beta} z + \bar{\alpha}\beta\bar{w} + |\beta|^2)^k}$$

since $|\alpha| = 1$

$$= e^{Lz\bar{w} + L\alpha\bar{\beta}z + L\bar{\alpha}\beta\bar{w} + L|\beta|^2}$$

$$= e^{\frac{1}{2}L|\beta|^2 + L\alpha\bar{\beta}z} \cdot \mathbb{E}\left[F_{\mathbb{C}}(z) \overline{F_{\mathbb{C}}(w)} \right] \cdot \overline{e^{\frac{1}{2}L|\beta|^2 + L\alpha\bar{\beta}w}}$$

$e^{Lz\bar{w}} = K_{F_{\mathbb{C}}}(z,w)$

$$= \mathbb{E}\Big[\underbrace{e^{\frac{1}{2}L|\beta|^2 + L\alpha\bar{\beta}z} F_{\mathbb{C}}(z)}_{=: H(z) \text{ GAF on } \mathbb{C}} \cdot \overline{e^{\frac{1}{2}L|\beta|^2 + L\alpha\bar{\beta}w} F_{\mathbb{C}}(w)} \Big]$$

$$= K_H(z,w)$$

Remark 2

$$\Longrightarrow G \overset{d}{=} H$$

H differs from $F_{\mathbb{C}}$ by a multiplication with a nowhere vanishing function

$$\Longrightarrow \{z \in \mathbb{C} \mid G(z) = 0\} \overset{d}{=} \{z \in \mathbb{C} \mid H(z) = 0\} = \{z \in \mathbb{C} \mid F_{\mathbb{C}}(z) = 0\} \qquad \square$$

First intensity measure

What remains to do for today is to show that $F_{\mathbb{C}}$ and $F_{\mathbb{D}}$ are essentially the only GAFs on $\mathbb{C}$ and $\mathbb{D}$ once we ask for the invariance property like in the previous proposition. Let's take a closer look at the zero set of a GAF.

in words

If F is a GAF on Λ then its zero set defines a point process on Λ through
$$\mu_F(\cdot) := \underbrace{\sum_{z \in F^{-1}(0)}}_{\text{zeros of } F} (\text{multiplicity of } z) \cdot \underbrace{\delta_z(\cdot)}_{\text{Dirac measure at } z}$$

random integer valued Borel measure on Λ which is finite on compact sets

The first thing you can ask about a point process is how many points are going to lie in a set A on average, this is called the first intensity measure of the point process.

in words

Definition 3: The first intensity measure of the point process μ_F is the (deterministic) measure on $\mathcal{B}(\Lambda)$ given by $\mathbb{E}[\mu_F(A)] = \mathbb{E}\left[\#\left\{\begin{matrix}\text{zeros of } F \text{ lying in } A \\ \text{with multiplicities}\end{matrix}\right\}\right] \quad \forall A \in \mathcal{B}(\Lambda)$

Fact : (we will see this next week)

For a GAF F on Λ one has the expression

$$\mathbb{E}[\mu_F(A)] = \int_A \frac{1}{4\pi} \Delta \log \underbrace{K(z,z)}_{=\mathbb{E}[|F(z)|^2]\geq 0 \text{ real number}} \underbrace{d\lambda(z)}_{\text{Lebesgue measure}}$$

interpreted in distributional sense

Remark 4: • $\log K(z,z)$ not defined when

$K(z,z) = \mathbb{E}[|F(z)|^2] = 0 \iff F(z) = 0$ a.s. $\iff$ z is deterministic zero of F

in these points the first intensity measure has an atom, i.e. $\mathbb{E}[\mu_F(\{z\})] =$ multiplicity of deterministic zero z > 0

• If z is not a deterministic zero of F

then $\mathbb{E}[\mu_F(\{z\})] = \sum_{k=0}^{\infty} k \, \mathbb{P}[F(z) = 0 \text{ , multiplicity is } k] = 0$

zero probability since $F(z)$ non-degenerate Gaussian since z not deterministic zero

• Our examples $F_{\mathbb{C}}$, $F_{\mathbb{D}}$ don't have deterministic zeros, indeed $K_{F_{\mathbb{C}}}(z,z) = e^{L|z|^2} > 0 \quad \forall z \in \mathbb{C}$

$K_{F_{\mathbb{D}}}(z,z) = (1-|z|^2)^{-L} > 0 \quad \forall z \in \mathbb{D}$

Examples: (the ones we care about)

- For $F_{\mathbb{C}}$, $\frac{1}{4\pi}\Delta \log \underbrace{K(z,z)}_{e^{L|z|^2}} \overset{!}{=} \frac{L}{\pi} \quad \forall z \in \mathbb{C}$ $\qquad \Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$

$$\Rightarrow \mathbb{E}[\mu_{F_{\mathbb{C}}}(A)] = \mathbb{E}[\#\{\text{zeros of } F_{\mathbb{C}} \text{ in } A\}] = \int_A \frac{L}{\pi}\, d\lambda(z)$$

$$= \frac{L}{\pi} \cdot |A|_{Leb}$$

"the first intensity measure of $F_{\mathbb{C}}$ is proportional to the Lebesgue measure on $\mathbb{C}$"

- For $F_{\mathbb{D}}$, $\frac{1}{4\pi}\Delta \log \underbrace{K(z,z)}_{(1-|z|^2)^{-L}} = \dots = \frac{L}{\pi} \cdot \frac{1}{(1-|z|^2)^2} \quad \forall z \in \mathbb{D}$

$$\Rightarrow \mathbb{E}[\mu_{F_{\mathbb{D}}}(A)] = \mathbb{E}[\#\{\text{zeros of } F_{\mathbb{D}} \text{ in } A\}] = \frac{L}{\pi}\int_A \underbrace{\frac{1}{(1-|z|^2)^2}\, d\lambda(z)}_{\substack{\text{hyperbolic area} \\ \text{measure w.r.t.} \\ \text{Lebesgue measure} \\ \text{on } \mathbb{D}}}$$

$$= \frac{L}{\pi} |A|_{\substack{\text{hyperbolic} \\ \text{measure}}}$$

"the first intensity measure of $F_{\mathbb{D}}$ is proportional to the hyperbolic area measure on $\mathbb{D}$"

"Uniqueness" of GAF

We are ready to show that two GAFs on the same domain $\Lambda \subseteq \mathbb{C}$ such that their zero sets form ~~[illegible]~~ point processes with same first intensity measures are essentially unique. In particular, it will imply that the first intensity measure of the zero set of a GAF completely determines the distribution of the zero set.

in words

Theorem: (Calabi's rigidity)

Suppos F, G are GAFs on Λ such that

$\mathbb{E}[\mu_F(\cdot)] = \mathbb{E}[\mu_G(\cdot)]$ (first intensity measures of zero sets are equal).

Then $F(\cdot) \overset{d}{=} \varphi(\cdot)\, G(\cdot)$ where φ is deterministic, analytic function on Λ without any zeros.

In particular, $\{z \in \Lambda \mid F(z) = 0\} \overset{d}{=} \{z \in \Lambda \mid G(z) = 0\}$

Proof: $\underline{1^{st} \text{ step}}$: Forget about deterministic zeros of F and G.

If z_0 deterministic zero of F, then by Remark 4

$$0 < \underbrace{\mathbb{E}[\mu_F(\{z_0\})]}_{\text{multipl. of } z_0 \text{ for } F} \underset{\substack{\uparrow \\ \text{assump.}}}{=} \underbrace{\mathbb{E}[\mu_G(\{z_0\})]}_{\text{multiplicity of } z_0 \text{ for } G}$$

$\overset{\text{Remark 4}}{\Longrightarrow}$ z_0 deterministic zero of G and multiplicity of z_0 for F and G is the same (and vice versa).

Let $D = \{\text{deterministic zeros of } F \text{ (and } G)\}$.

Assume we show Theorem for domain $\Lambda \setminus D$,

i.e. $F\big|_{\Lambda \setminus D} \overset{d}{=} \tilde{\varphi} \cdot G\big|_{\Lambda \setminus D}$ with $\tilde{\varphi}$ deterministic, analytic function on $\Lambda \setminus D$ and non-vanishing

We claim that $\tilde{\varphi}$ can be extended analytically to any $z_0 \in D$. Indeed, in some neighbowhood of z_0 for some k

$$F(z) = \cancel{(z-z_0)^k} \underbrace{\tilde{F}(z)}_{\substack{\text{analytic with} \\ \tilde{F}(z_0) \neq 0}} \overset{d}{=} \tilde{\varphi}(z) \cdot \cancel{(z-z_0)^k} \underbrace{\tilde{G}(z)}_{\substack{\text{analytic with} \\ \tilde{G}(z_0) \neq 0}}$$

since z_0 has same multiplicity k for F and G

$\Rightarrow$ $\tilde{\varphi}$ bounded in this neighbourhood ("$\tilde{G}$ could not eat up some unboundedness of $\tilde{\varphi}$ since $\neq 0$ in and around z_0")

$\Rightarrow$ $\tilde{\varphi}$ extends analytically to z_0

$\Rightarrow$ $F \overset{d}{=} \varphi \cdot G$ on Λ as claimed

(φ non-vanishing in any $z_0 \in D$ because z_0 has same multiplicity for F and G)

<u>2nd step</u>: Show statement assuming F (and G) have no deterministic zero.

By Remark 4, $K_F(z,z) \neq 0$ and $K_G(z,z) \neq 0$ $\forall z \in \Lambda$.

$$\Rightarrow \mathbb{E}[\mu_F(A)] = \int_A \frac{1}{4\pi} \Delta \log K_F(z,z) \, d\lambda(z) \qquad \forall A \in \mathcal{B}(\Lambda)$$

$$\mathbb{E}[\mu_G(A)] = \int_A \frac{1}{4\pi} \Delta \log K_G(z,z) \, d\lambda(z)$$

$$\Rightarrow \Delta \log K_F(z,z) = \Delta \log K_G(z,z) \qquad \forall z \in \Lambda$$

$$\Rightarrow u(z) := \underbrace{\log K_F(z,z)}_{\text{recall this is a real function}} - \log K_G(z,z)$$ is harmonic function ($\Lambda \subseteq \mathbb{C}$ seen as subset of $\mathbb{R}^2$)

$$\Rightarrow K_F(z,z) = e^{u(z)} K_G(z,z)$$

Assume Λ is simply connected (as in our main cases $\mathbb{C}$, $\mathbb{D}$). Then every (real) harmonic function on Λ (seen as subset of $\mathbb{R}^2$) can be written as the real part of an analytic function on Λ (this is an easy fact from complex analysis)

$\Rightarrow \exists\, \Psi : \Lambda \to \mathbb{C}$ analytic with $u(z) = 2\,\mathrm{Re}(\Psi(z))$

Set $\varphi(\cdot) := e^{\Psi(\cdot)}$ analytic on Λ
non-vanishing

$$\Rightarrow K_F(z,z) = e^{u(z)} K_G(z,z)$$
$$= e^{2\mathrm{Re}(\Psi(z))} K_G(z,z)$$
$$= e^{\Psi(z) + \overline{\Psi(z)}} K_G(z,z)$$
$$= \varphi(z)\overline{\varphi(z)}\, K_G(z,z) \qquad (*)$$

Consider function $f(z,w) = K_F(z,w) - \varphi(z)\overline{\varphi(w)} K_G(z,w)$

$\Rightarrow$ f is analytic in z on $\Lambda \times \Lambda$
f is anti-analytic in w (i.e. analytic in $\overline{w}$)

$K_F(z,w) = \mathbb{E}[F(z)\overline{F(w)}]$
and then use that F is analytic, the same holds for $K_G(z,w)$

moreover $f(z,z) = 0 \ \forall z \in \Lambda$ by $(*)$

$\Rightarrow f \equiv 0$, i.e. $K_F(z,w) = \varphi(z)\overline{\varphi(w)} K_G(z,w)$

$$= \mathbb{E}\left[\varphi(z) G(z) \overline{\varphi(w)}\,\overline{G(w)}\right]$$
$$= K_{\varphi\cdot G}(z,w) \qquad \forall z,w$$

result from complex analysis, maybe we see it at the end

$\xRightarrow{\text{Remark 2}} f \overset{d}{=} \varphi \cdot G$ as requested

Assume Λ is not simply connected.

Fix $z_0 \in \Lambda$, $r > 0$ with $D_{z_0}(r) \subseteq \Lambda$.

$\overset{\text{previous situation}}{\Longrightarrow}$ $\exists\ \tilde{\varphi} : D_{z_0}(r) \to \mathbb{C}$ analytic, non-vanishing with $K_F(z,w) = \tilde{\varphi}(z)\overline{\tilde{\varphi}(w)}\, K_G(z,w)$ $\forall z,w \in D_{z_0}(r)$ (**)

Note that for any $w \in D_{z_0}(r)$ the function $\vartheta_w(z) = \dfrac{K_F(z,w)}{\overline{\tilde{\varphi}(w)}\, K_G(z,w)}$ is analytic on $\Lambda \setminus \{z \in \Lambda \mid K_G(z,w) = 0\}$ and equals $\tilde{\varphi}$ on $D_{z_0}(r)$ by (**).

Moreover for $w, w' \in D_{z_0}(r)$ these functions $\vartheta_w, \vartheta_{w'}$ are equal on the set $D_{z_0}(r)$ and hence they are equal on the whole of $\Lambda \setminus \{z \in \Lambda \mid K_G(z,w) = 0 = K_G(z,w')\}$ where they both are analytic (since $D_{z_0}(r)$ contains limit points).

Hence φ defined on $\Lambda' := \Lambda \setminus \{z \in \Lambda \mid K_G(z,w) = 0\ \forall w \in D_{z_0}(r)\}$ by $\varphi(z) := \vartheta_w(z)$ for some $w \in D_{z_0}(r)$ with $K_G(z,w) \neq 0$ is well-defined, deterministic, analytic and $\varphi|_{D_{z_0}(r)} = \tilde{\varphi}|_{D_{z_0}(r)}$.

But $\Lambda' = \Lambda$, else $K_G(z,\cdot)$ would be anti-analytic function vanishing on $D_{z_0}(r)$ and thus zero everywhere ($D_{z_0}(r)$ contains limit point) $\lightning$

$$\Longrightarrow \varphi(z) = \vartheta_w(z) = \frac{K_F(z,w)}{\overline{\tilde{\varphi}(w)}\, K_G(z,w)} \overset{w \in D_{z_0}(r)}{=} \frac{K_F(z,w)}{\overline{\varphi(w)}\, K_G(z,w)}$$

$\forall z \in \Lambda$ and the corresponding w

$$\Rightarrow \quad K_F(z,w) = \varphi(z)\overline{\varphi(w)}\, K_G(z,w) \qquad \forall z \in \Lambda$$
$$\forall w \in D_{z_0}(r) \text{ with } K_G(z,w) \neq 0$$

$\Rightarrow$ this relation has to hold for all $z, w \in \Lambda$, too, since both sides are anti-analytic in w on Λ and agree on $\underbrace{\{w \in D_{z_0}(r) \mid K_G(z,w) \neq 0\}}_{\text{contains limit point, else } K_G(z,w) \equiv 0}$

Remark 2

$\Rightarrow \quad F \overset{d}{=} \varphi \cdot G$ with φ analytic on Λ

$\Rightarrow \quad \frac{1}{\varphi}$ analytic on Λ, hence non-vanishing

exchange roles of F, G

$\Rightarrow \quad F \overset{d}{=} \varphi \cdot G$ with φ as requested

In particular $\{z \in \Lambda \mid F(z) = 0\} \overset{d}{=} \{z \in \Lambda \mid G(z) = 0\}$ since φ non-vanishing.

$\square$

Corollary: If G is GAF on $\mathbb{C}$ resp. $\mathbb{D}$ such that the zero set $G^{-1}(0)$ is invariant in distribution under the euclidean resp. hyperbolic isometries of $\mathbb{C}$ resp. $\mathbb{D}$, then $\exists\, \varphi: \mathbb{C}, \mathbb{D} \to \mathbb{C}$ deterministic, analytic, nowhere vanishing with $G \overset{d}{=} \varphi \cdot F_{\mathbb{C}}$ resp. $G \overset{d}{=} \varphi \cdot F_{\mathbb{D}}$ (for some parameter $L>0$ in $F_{\mathbb{C}}, F_{\mathbb{D}}$).

"$F_{\mathbb{C}}$, $F_{\mathbb{D}}$ are unique if we ask for the invariance"

Proof: Once we show that the first intensity measures of G and $F_{\mathbb{C}}$ resp. of G and $F_{\mathbb{D}}$ are equal we are done by the previous theorem.

Note that for $A \in \mathcal{B}(\mathbb{C})$ resp. $\mathcal{B}(\mathbb{D})$

$$\underbrace{\mathbb{E}[\mu_G(A)]}_{\text{first intensity measure of } G \text{ of set } A} = \mathbb{E}[\#\{\text{zeros of } G \text{ in } A\}]$$

has to be a scaling of the Lebesgue measure resp. hyperbolic area measure of A since these are the unique measures (up to scaling) on $\mathbb{C}$ resp. $\mathbb{D}$ which are invariant under the resp. isometries.

$$\Rightarrow \quad \mathbb{E}[\mu_G(\cdot)] = c|\cdot|_{\text{Lebesgue}} \quad \text{resp.} \quad \mathbb{E}[\mu_G(\cdot)] = c|\cdot|_{\text{hyperb.}}$$

Choose $L>0$ such that $\frac{L}{\pi} = c$

$$\underset{\text{see Examples}}{\Rightarrow} \quad \mathbb{E}[\mu_G(\cdot)] = \mathbb{E}[\mu_{F_{\mathbb{C}}}(\cdot)] \quad \text{resp.} \quad \mathbb{E}[\mu_G(\cdot)] = \mathbb{E}[\mu_{F_{\mathbb{D}}}(\cdot)]$$

$\square$