Notations

open connected domain of $\mathbb{C}$

F is a GAF on Λ, if $z_1, \dots, z_n \in \Lambda$ $(F(z_1), \dots \dots F(z_n))$ Gaussian vector.

for $z, w \in \Lambda$

$$k_F(z,w) := \mathbb{E}(f(z)\overline{f(w)})$$ the covariance kernel.

if $0 \in \Lambda$
near 0. f has the expansion

$$f(z) = \sum_{n\geq 0} a_n z^n$$

$$k_F(z,w) = \sum \underbrace{\mathbb{E}(a_n \overline{a_m})}_{a_{n,m}} z^n \cdot \overline{w}^m$$

$\Rightarrow$ $k(z,z)$ continuous

$\Leftrightarrow$ k is analytic in z anti analytic in w

Zeros of f

$$\mu_F(\cdot) = \sum_{x \in F^{-1}(0)} \delta_x(\cdot)$$ Random Radon measure. counted with multiplicity

$\mathbb{E}(\mu_F(\cdot))$ is the first intensity measure of $F^{-1}(0)$

$\xi \sim \mathcal{N}_{\mathbb{C}}(0,1)$ then $|\xi|^2 \sim Exp(1)$

$$P_{\mathcal{N}_{\mathbb{C}}(0,1)}(z) \propto e^{-|z|^2}$$

Example. GAF

$$F_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n$$ defined on $\mathbb{C}$ $\xi_n \sim$ iid $\mathcal{N}_{\mathbb{C}}(0,1)$, $L > 0$.

$$k_{F_{\mathbb{C}}}(z,w) = \exp(L z\overline{w})$$

$$\rho_{\mathbb{C}}(z) = \frac{L}{\pi}$$

First intensity $\rightsquigarrow$ Study $\mathbb{E}(\mu_f(\cdot))$

1) Deterministic analytic function.

Lemma. 1 $\forall \varphi \in C_c^\infty(\Delta)$

$$\int_\Delta \Delta\varphi(z) \cdot \underbrace{\frac{1}{2\pi} \log|z - z_0|}_{\text{Green's function for } \Delta \; G(z, z_0)} \underbrace{d\,\mathrm{vol}(z)}_{\text{Lebesgue}} = \varphi(z_0)$$

$\Leftrightarrow$ in the distributional sense

" $\Delta_z G(z, z_0) = \delta_{z_0}$ "

" $\Delta G(\cdot, z_0) = \delta_{z_0}(\cdot)$ "

Prop 2: For f analytic on Δ. $\forall \varphi \in C_c^\infty(\Delta)$

$$\int_\Delta \Delta\varphi(z) \frac{1}{2\pi} \log|f(z)| \, d\,\mathrm{vol}(z) = \mu_f(\varphi)$$

$\Leftrightarrow$ Distributional sense

" $\Delta\left(\frac{1}{2\pi} \log|f(\cdot)|\right) = \mu_f(\cdot)$ "

Proof: Let $\varphi \in C_c^\infty(\Delta)$ $\quad$ $\mathrm{Supp}(\varphi)$ compact

zeros of f are isolated $\Rightarrow \#\left(f^{-1}(0) \cap \mathrm{Supp}(\varphi)\right) < \infty$.

$\exists g(z)$ s.t. $f(z) = g(z) \prod_k (z - z_k)^{m_k}$

where $z_k \in f^{-1}(0) \cap \mathrm{Supp}(\varphi)$

m_k its multiplicity.

$g \neq 0$ on $\mathrm{Supp}(\varphi)$

$$\log|f(z)| = \underbrace{\log|g(z)|}_{\text{harmonic on } \mathrm{supp}(\varphi)} + m_k \log|z - z_k|$$

so $\int_\Lambda \Delta\varphi \frac{1}{2\pi} \log|f(z)|\, d\mathrm{vol}(z)$

$$= \int_\Lambda \Delta\varphi \frac{1}{2\pi}\left(\log|g(z)| + m_k \log|z - z_k|\right) d\mathrm{vol}(z)$$

$$= \int_\Lambda \varphi\, \Delta\left(\frac{1}{2\pi}\log|g(z)|\right) d\mathrm{vol}(z) \quad \leftarrow 0$$

$$+ \sum m_k\, \varphi(z_k))$$

↑ from Lemma 1

$$= \mu_f(\varphi)$$ ∎

2) Now let F be GAF on $\Lambda \ni 0$

Lemma 3: $\boxed{F(z) = \sum_{n\geq 0} a_n z^n}$ in a neighbourhood of 0

i) $F^{(n)}$ are GAF.

ii) Then a_n are centered complex Gaussian.

$$= \frac{F^{(n)}(0)}{n!}$$

iii) $\forall z_0 \in \Lambda$, $F(z_0) = 0$ has probability 0 or 1 as well as its multiplicity. (deterministic)

and $K(z, z) = |z - z_0|^{2m} L(z, z)$

where $L(z, z)$ is non-zero, m = multiplicity of z_0.

Proof. i) For $n = 1$

$$F'(z) = \frac{F(z+\varepsilon) - F(z)}{\varepsilon} \leftarrow \text{centered Gaussian}$$

$\xrightarrow[\varepsilon\to 0]{}$ converges a.s.

$(F'(z_1), F'(z_2), \cdot F(z_n))$ is Gaussian vector. limit is also Gaussian.

ii) Consequence of i)

iii) $F(z_0) \sim \mathcal{N}_{\mathbb{C}}^{1}(0, k(z_0, z_0))$

either $k(z_0, z_0) = 0 \Leftrightarrow \mathbb{P}(F(z_0) = 0) = 1$

$k(z_0, z_0) \neq 0 \Leftrightarrow \mathbb{P}(F(z_0) = 0) = 0$

z_0 is a zero with multiplicity m of F

iff $F(z_0) = 0, \; F'(z_0) = 0, \ldots, F^{(m-1)}(z_0) = 0.$

$\mathbb{P} \in \{0, 1\}$

Assume z_0 is zero of F with multiplicity m deterministic

Then $\dfrac{F(z)}{(z - z_0)^m} = H(z)$ GAF. with $H(z_0) \neq 0$ a.s

$$K_F(z, z) = |z - z_0|^{2m} \underbrace{\mathbb{E}(H(z)\overline{H(z)})}_{L(z,z)}$$

$\boxtimes$

$\Rightarrow$ Allows us to prove the following theorem about the first intensity.

Thm 4 (Edelman - Kostlan Formula)

$\forall \varphi \in C_c^\infty(\Lambda)$

$$\mathbb{E}(\mu_F(\varphi)) = \int_\Lambda \Delta\varphi(z) \frac{1}{4\pi} \log k(z, z) \, d\mathrm{vol}(z)$$

"
$$\mathbb{E}(\mu_F(\cdot)) = \frac{1}{4\pi} \Delta \log k(z, z) \, d\mathrm{vol}(z) \text{"}$$

Consequence $\mathbb{E}(\mu_F(\cdot))$ is locally finite.

$\Rightarrow \sigma$- finite measure.

In fact we have seen that (Lemma 3)

$$k(z,z) = |z-z_0|^{2m} L(z,z) \quad \text{for } m \geq 1,\ m \in \mathbb{N}$$

if $k(z_0, z_0) = 0$

So in a small neighborhood of z_0

$$\log k(z,z) = 2m \log|z-z_0| + \underbrace{\log L(z,z)}_{C^\infty}$$

$$\Delta \log k(z,z) = 2m \cdot 2\pi\, \delta_{z_0} + \underbrace{(\cdots)}_{C^\infty}$$

$\Rightarrow$ locally finite.

Proof of Thm 4.

Prop 2 $\Rightarrow$ $\mathbb{E}(\mu_F(\varphi)) = \mathbb{E}\left(\int_\Delta \Delta\varphi(z) \frac{1}{2\pi} \log|F(z)|\, d\mathrm{vol}(z)\right)$

I would like to exchange $\mathbb{E}$ and $\int$. ok ✓

Since $\int_\Delta |\Delta\varphi(z)| \frac{1}{2\pi} \mathbb{E}\left|\log|F(z)|\right| d\mathrm{vol}(z)$

$$= \int_{\mathrm{supp}(\varphi)} |\Delta\varphi(z)| \frac{1}{2\pi} \left(\overbrace{\mathbb{E}\left|\log|\xi|\right|}^{\infty} + \underbrace{\left|\log\sqrt{k(z,z)}\right|}_{\log\left(|z-z_0|^{2p}\right) + \log(C^\infty)}\right) d\mathrm{vol}(z)$$

$\xi \sim N_{\mathbb{C}}(0,1)$

$< \infty$

$$= \int_\Delta \Delta\varphi(z) \frac{1}{2\pi} \mathbb{E}\left(\log|F(z)|\right) d\mathrm{vol}(z)$$

$$= \int_\Delta \Delta\varphi(z) \frac{1}{2\pi} \left(\underbrace{\mathbb{E}\left(\log|\xi|\right)}_{\substack{\Delta\ \text{constant} \\ = 0}} + \log\sqrt{k(z,z)}\right) d\mathrm{vol}(z)$$

$$= \int_\Delta \Delta\varphi(z) \frac{1}{4\pi} \log k(z,z)\, d\mathrm{vol}(z)$$

▨

Physicist's approach. C. Rigidity is not constructive. But in special cases. it works.

From the translation invariance $\to$ back to function $F_{\mathbb{C}}$

If f. GAF on $\mathbb{C}$. with $\mathbb{E}(\mu_f(\cdot))$

The first intensity measure is σ-finite

+ translation invarrant

on $\mathcal{B}(\mathbb{C})$

$\Rightarrow$ it is a multiple of Lebesgue measure.

assume: $\mathbb{E}(\mu_f(A)) = \dfrac{Leb(A)}{\pi}$ no atom $\Rightarrow k(z,z) \neq 0, \forall z$.

$$\Rightarrow \frac{1}{\pi} = \frac{1}{4\pi} \Delta \log k(z.z) \qquad \Delta = 4\partial\bar{\partial}$$

$$\Rightarrow \partial\bar{\partial} \log k(z.z) \equiv 1$$

$$\log k(z,w) := \sum_{n.m\geq 0} b_{n.m}\, z^n \bar{w}^m$$

$$\log k(z,z) = \sum_{n.m\geq 0} b_{n,m}\, z^n \bar{z}^m$$

$$\partial\bar{\partial}\log k(z,z) = \sum_{n,m\geq 1} b_{n.m}\; n.m\; z^{n-1}\bar{z}^{m-1}$$

$$\equiv 1.$$

$\Rightarrow b_{n.m} = 0$ for $n,m \geq 1$. and $(n,m) \neq (1,1)$

$b_{1,1} = 1$ don't know about $b_{k,0}$

$$\mathbb{R} \ni \log k(z,z) = \sum_{k\geq 1} b_{0,k}\, \bar{z}^k + \sum_{k\geq 1} b_{k,0}\, z^k + z\bar{z} + \underset{\substack{\cap \\ \mathbb{R} \\ k(0,0)}}{c}$$

$\Rightarrow b_{0,k} = \overline{b_{k,0}} \Rightarrow \log k(z,z) = 2\,\mathrm{Re}(\sum b_k\, z^k) + z\cdot\bar{z}$

$$k(z,z) = \exp\left(2\,\mathrm{Re}\left(\sum b_{k,0} z^k\right) + z\bar{z}\right) \cdot k(0,0)$$

$$= \exp\left(\sum b_{k,0} z^k + \sum b_{0,k} \bar{z}^k + z\bar{z}\right) \cdot k(0,0)$$

$\mathbb{E}\left(f(z)\overline{f(z)}\right)$

$$\varphi(z) = \exp\left(\sum b_{k,0} z^k\right) \sqrt{k(0,0)}$$

no zero.

when $b_{k,0} \equiv 0$

$$\exp(z\bar{z}) = \sum_{n \geq 0} \frac{1}{n!} z^n \bar{z}^n$$

$$\Rightarrow \quad \mathbb{E}(a_n \cdot \bar{a}_m) = \delta_{n,m} \frac{1}{n!}$$

$$\left(a_n \cdot \sqrt{n!}\right) \sim \text{iid } \mathcal{N}^1_{\mathbb{C}}(0,1)$$

$\Rightarrow \varphi \cdot F_{\mathbb{C}}(z)$ with $L = 1$ works

...rties of GAF
Simple p.p

Lemma. Let f be a GAF in Λ without deterministic zero, then a.s f has no zeros of multiplicity ≥ 1.

I.e. $f^{-1}(0)$ is a simple point process apart from the deterministic zeros.

Proof: We show that a.s. there is no z s.t $f(z)=f'(z)=0$.

Fix $z_0 \in \Lambda \Rightarrow k(z_0, z_0) \neq 0$

- Define $h(z) = f(z) - \dfrac{k(z, z_0)}{k(z_0, z_0)} f(z_0)$

$$\mathrm{Cov}\big(h(z), f(z_0)\big) = k(z, z_0) - \frac{k(z, z_0)}{k(z_0, z_0)} k(z_0, z_0)$$

$$= 0$$

so $h(z)$ and $f(z_0)$ is independent $\forall z \in \Lambda$

h is also GAF.

- For z s.t $k(z, z_0) \neq 0$

$$\frac{f(z)}{k(z, z_0)} = \frac{h(z)}{k(z, z_0)} \overset{\perp\!\!\!\perp}{+} \frac{f(z_0)}{k(z_0, z_0)}$$

If z multiple zero of f.

Two cases.

- z is a multiple zero of RHS ①
- $k(z, z_0) = 0$ (Formula not valuable) ②

② Zeros of $k(\cdot, z_0)$ is a deterministic set and are isolated $\Rightarrow$ countable.

for z_1 s.t $k(z_1, z_0) = 0$. $f(z_1) = 0$ with prob 1 or 0.

multiple zero with pb 1 or 0.

Property of GAF 2 simple pp.

① If z is multiple zero of RHS and $k(z, z_0) \neq 0$

$$\left(\frac{h(z)}{k(z,z_0)}\right)'(z) = 0$$

and

$$\underbrace{\frac{f(z_0)}{k(z_0,z_0)}}_{\substack{\text{Non degenerate} \\ \text{Gaussian conditioned on } h(\cdot)}} = -\frac{h(z)}{k(z,z_0)}$$

let S be the random set $\sigma(h)$-meas

$$\left\{ z : \left(\frac{h(z)}{k(z,z_0)}\right)'(z) = 0 \right\}$$

S is countable discrete.

$$\mathbb{P}\left(\frac{f(z_0)}{k(z_0,z_0)} \in \underbrace{\left\{ -\frac{h(z)}{k(z,z_0)} ; z \in S \right\}}_{\substack{\text{countable random subset of } \mathbb{R} \\ \text{independent of } f(z_0)}} \right)$$

$= 0$

$\square$

Rq. The proof doesn't use the fact that $\mathbb{E}(f(z)) = 0$

So. $f - w$ has no non-deterministic multiple zeros.

if $w \neq 0$. and no deterministic zero neither

as $\mathbb{E}(f(z) - w) = -w$.

Ergodicity 1

Déf: Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space $G \circlearrowright \Omega$ (Group)

preserving measure $\mathbb{P}$

i.e. $\mathbb{P}(\tau^{-1}(\cdot)) = \mathbb{P}(\cdot) \quad \forall \tau \in G$.

A is an <u>invariant event</u> if $\tau(A) = A$.

The action of G is ERGODIC if

$$\mathbb{P}(A) \in \{0,1\} \quad \forall A \text{ invariant event.}$$

We say that $\mathbb{P}$ is ergodic under the action of G.

Example here: $\Omega = \mathcal{M}(\Lambda)$

σ-finite measures

is complete separable metric space.

Radon metric

$\Rightarrow$ gives weak convergence.

$\mathcal{A}$. Borel.

$\mathbb{P}$. given by zero measure of $F_{\mathbb{C}}$

$G = \mathrm{Isom}_{\mathbb{C}} = \{ z \mapsto \lambda z + \beta \ ; \ |\lambda| = 1 \}$

$\left\{ \lim_{a \to \infty} \frac{1}{4a^2} \mu_f([-a, a]^2) = c \right\}$ is an invariant event.

Prop: The zero set of $F_{\mathbb{C}}$ ($F_{\mathbb{D}}$) is ergodic under $\mathrm{Isom}(\mathbb{C})$. ($\mathrm{Isom}(\mathbb{D})$)

Consequence, the density of zeros of $F_{\mathbb{C}}$ is deterministic.