Ergodicity 2

Proof for $F_{\mathbb{C}}$. $L=1$

$$F_{\mathbb{C}}(z) = \sum a_n \frac{1}{\sqrt{n!}} z^n.$$

For $\beta \in \mathbb{C}$. $f_\beta = f(z+\beta) \cdot e^{-z\bar{\beta} - \frac{1}{2}|\beta|^2}$

We have seen last week that $f_\beta \overset{d}{=} f$ when we prove the invariance of zero set of f.

$$\mathbb{E}(f_\beta(z)\bar{f}(w)) = e^{-z\bar{\beta} - \frac{1}{2}|\beta|^2} K(z+\beta, w)$$
$$= e^{-z\bar{\beta} - \frac{1}{2}|\beta|^2 + z\bar{w} + \beta\bar{w}}$$

when $\beta \to \infty$ $\quad \mathbb{E}(f_\beta(z)\bar{f}(w)) \xrightarrow[\text{unif. in compacts } z,w \in K.]{} 0$

which means that $f_\beta(z)$ and $\bar{f}(w)$ are asymptotically independent (Gaussian process)

Let A be an invariant event

A_K be an event that depending only on distribution of zeros on a compact set $K \subset \Lambda$

$\sigma(A_K) = \mathcal{A}$ ← Borel σ-algebra generated

$\forall \varepsilon > 0. \quad \exists A_K$ s.t $\mathbb{P}(A \Delta A_K) \leq \varepsilon$.

$$\left| \mathbb{E}[\mathbb{1}_A(\mu_f)\mathbb{1}_A(\mu_{f_\beta})] - \mathbb{E}(\mathbb{1}_{A_K}(\mu_f)\mathbb{1}_{A_K}(\mu_{f_\beta})) \right|$$
$$\leq |\mathbb{E}[\mathbb{1}_A(\mu_f)|\mathbb{1}_A(\mu_{f_\beta}) - \mathbb{1}_{A_K}(\mu_{f_\beta})|]$$
$$+ \mathbb{E}[|\mathbb{1}_A(\mu_f) - \mathbb{1}_{A_K}(\mu_f)|\,\mathbb{1}_{A_K}(\mu_{f_\beta})]$$

Ergodicity 3

$$\leq \mathbb{E}\left[\left|\mathbb{1}_A(\mu_{f_\beta}) - \mathbb{1}_{A_k}(\mu_{f_\beta})\right|\right] + \mathbb{E}\left[\left|\mathbb{1}_A(\mu_f) - \mathbb{1}_{A_k}(\mu_f)\right|\right]$$

$$\leq 2\varepsilon.$$

By assymptotic independence of f and f_β on K.

$$\mathbb{E}\left[\mathbb{1}_{A_K}(\mu_f)\, \mathbb{1}_{A_K}(\mu_{f_\beta})\right] \xrightarrow{\beta\to\infty} \mathbb{E}\left[\mathbb{1}_{A_k}(\mu_f)\right]\mathbb{E}\left(\mathbb{1}_{A_k}(\mu_f)\right)$$

$$= \mathbb{E}\left(\mathbb{1}_{A_k}(\mu_f)\right)^2$$

We have also $\left|\left[\mathbb{E}(\mathbb{1}_{A_k}(\mu_f))\right]^2 - \left[\mathbb{E}(\mathbb{1}_{A}(\mu_f))\right]^2\right| \leq 2\varepsilon$

$$\overline{\lim_{\beta\to\infty}} \left|\mathbb{E}\left(\mathbb{1}_A(\mu_f)\, \mathbb{1}_A(\mu_{f_\beta})\right) - \left[\mathbb{E}\left(\mathbb{1}_A(\mu_f)\right)\right]^2\right| \leq 4\varepsilon$$

$$\Rightarrow \quad \mathbb{E}\left(\mathbb{1}_A(\mu_f)\, \mathbb{1}_A(\mu_{f_\beta})\right) = \left[\mathbb{E}\left(\mathbb{1}_A(\mu_f)\right)\right]^2$$

since A is translation invariant

$$\mathbb{1}_A(\mu_{f_\beta}) \leftarrow \text{translation by } -\beta$$

$$= \mathbb{1}_A(\mu_f - \beta)$$

$$= \mathbb{1}_A(\mu_f)$$

$$\Rightarrow \quad \mathbb{E}\left(\mathbb{1}_A(\mu_f)\right) = \left[\mathbb{E}\left(\mathbb{1}_A(\mu_f)\right)\right]^2$$

$$\Rightarrow \quad \mathbb{E}\left(\mathbb{1}_A(\mu_f)\right) \in \{0, 1\}.$$

$\square$