Gravitational Allocation
to zeroes of the Gaussian entire function.

Reading group: Zürich, 2017.

1 Optimal Transport

1.1 The problem

The theory of optimal transport stems from the following problem proposed by Monge in 1781 [Mon81] and later studied by Kantorovich [Kan42] in the 1940s. Suppose we have two probability measures μ and ν on a domain $\Omega \subset \mathbb{R}^d$ and let $\Pi(\mu, \nu)$ be the set of *couplings* between μ and ν. In the optimal transport world these are known as *transport plans* from μ to ν, and represent a scheme for transporting mass distributed under μ to mass distributed under ν. We would like to know which transport plans are the most "cost-effective", in the sense that mass is moved as small a distance as possible.

Question 1.1 *What is*

$$\min\{\int |x-y|\pi(dx, dy) \,|\, \pi \in \Pi(\mu,\nu)\}? \tag{1.1}$$

And what are/is the minimising $\pi \in \Pi(\mu, \nu)$?

A transport plan is called a *transport map* if the coupling is deterministic (i.e. if (X, Y) has law π, then $X \sim \mu$, $Y \sim \nu$ and $Y = T(X)$ is a deterministic function of X). Equivalently, $\pi \in \Pi$ is a transport map if π is the image measure of μ under a map of the form $(id \times T) : \Omega \to \Omega \times \Omega$. The natural question then follows:

Question 1.2 *When do transport maps exist? And when are optimal transport plans given by transport maps?*

Note that the first question is non-trivial: if μ is a dirac mass, then there cannot exist a transport map from μ to anything other than another dirac mass.

Remark 1.3 *In fact, it can be shown that unique optimal transport plans do exist, and are given by transport maps, as long as μ is absolutely continuous with respect to Lebesgue measure. However, the minimal cost need not be finite.* [1]

From now on, we will be primarily interested in just constructing transport plans between measures (in a more general framework where the measures in question need not be absolutely continuous or finite). We will return to the problem of how costly they are later.

[1] For example if μ is an absolutely continuous probability measure on $\mathbb{R}^d$ with infinite mean and $\nu = \delta_0$, then the cost will always be infinite.

1.2 Moser's deformation scheme

Suppose we are working on the open unit disc $\mathbb{U} \subset \mathbb{R}^2$, and that we have two probability measures μ and ν on $\mathbb{U}$ that are absolutely continuous with respect to Lebesgue measure, and have smooth densities bounded strictly away from 0. Let X be a random variable with law μ. Given a realisation of X, it is not too hard to believe that we should be able to construct a continuous curve $\gamma : [0,1] \to \mathbb{U}$ (which will be a deterministic function of X) such that

(a) $\gamma(0) = X$; $\gamma(1) \overset{(d)}{=} \nu$; and

(b) $\dot{\gamma}(t) = v(t, \gamma(t))$ for all $t \in [0,1]$, where $v : [0,1] \times \mathbb{U} \to \mathbb{R}^2$ is a deterministic locally Lipschitz function, with Neumann boundary condition $\partial_n v(t,x) = 0$ for every $t \in [0,1]$ and $x \in \partial\mathbb{U}$.

The Neumann boundary condition here is natural since ν has zero mass on the boundary $\partial\mathbb{U}$, and we would like to $\gamma(1)$ to have law ν.

In fact, there are not enough constraints here to determine the function v (if we have a deterministic way to define $\gamma(1)$ from $\gamma(0) = X$, then there are still plenty of smooth curves we could draw between them). So, we add the natural assumption:

(c) $\mathrm{Law}(\gamma(t)) = \rho_t$ for all $t \in [0,1]$ where $\rho_t := (1-t)\mu + t\nu$.

In the following we also write $\mu(x), \nu(x)$ and $\rho_t(x)$ for the densities of μ, ν and ρ, which are all well defined by assumption.

Given (a), (b) and (c) observe that for any $\phi \in C_c^\infty(\mathbb{U})$

$$\frac{d}{dt}\mathbb{E}[\phi(\gamma(t))] = \frac{d}{dt}\big((1-t)\int_{\mathbb{U}} \phi(x)\mu(x)dx + t\int_{\mathbb{U}} \phi(x)\nu(x)dx\big) = \int_{\mathbb{U}} \phi(x)(\nu(x) - \mu(x))dx.$$

On the other hand,

$$\frac{d}{dt}\mathbb{E}[\phi(\gamma(t))] = \mathbb{E}[\nabla\phi \cdot \dot{\gamma}(t)] = \int_{\mathbb{U}} \nabla\phi \cdot v(t,x)\rho_t(x)\, dx = -\int_{\mathbb{U}} \phi(x)\big(\nabla \cdot v(t,x)\, \rho_t(x)\big)\, dx.$$

Putting these together we deduce that

$$-\big(\nabla \cdot v(t,x)\rho_t(x)\big) = \nu(x) - \mu(x) \tag{1.2}$$

for all $t \in [0,1]$ and $x \in \mathbb{U}$.

This motivates the following explicit construction of a transport between μ and ν, due to Moser [Mos65]. Let u be a solution of the equation

$$\Delta u(x) = \nu(x) - \mu(x) \tag{1.3}$$

with Neumann boundary conditions in $\mathbb{U}$ (such a u exists due to our smoothness assumptions on ν and μ). Define

$$v(t,x) := -\frac{\nabla u}{\rho_t(x)}$$

and for every $x \in \mathbb{U}$, let $Z_t(x)$ be the solution of the equation

$$\frac{dZ_t(x)}{dt} = v(t, Z_t(x)) \;\; ; \quad Z_0(x) = x. \tag{1.4}$$

Then it follows from exactly the same arguments as used to derive (1.2) that if X has law μ, then $Z_t(X)$ has law ρ_t for every $t \in [0,1]$. In particular $Z_1(X)$ is a deterministic function of X and has law ν: providing us with a nice explicit transport map between μ and ν.

Remark 1.4 *This is* not *an optimal transport. For optimality, it turns out that we would need the random curve γ to be a geodesic in $\mathbb{U}$ (i.e. a straight line) almost surely, and we can check that this is not true in the above construction (see [Vil08, Chapter 13]). However, it is still extremely nice to have a simple and explicit way to transport μ to ν.*

1.3 Dirac masses

Now it is clear that it is not possible to transport a unit dirac mass to anything other than another unit dirac mass. However, it *is* possible to transport from an absolutely continuous measure to a dirac mass (or collection of dirac masses); and this is exactly the type of transport we are interested in.

In this case, the above deformation scheme does not entirely make sense because (a) smooth solutions to (1.3) do not exist and (b) the density $\rho_t(x)$ does not exist for all $t \in [0,1]$. To get around (a) we instead search for weak solutions to (1.3), which will exist, for example, if μ is absolutely continuous and ν is a finite collection of point masses. To solve the problem (b) we simply remove $\rho_t(x)$ from the denominator, as it is a scalar and so only affects the speed at which $Z_t(x)$ reaches its final destination (the first atom of ν that it hits). If we then decide that $Z_t(x)$ should be constant after it hits this atom, and consider solutions of (1.4) on $[0,\infty)$ rather than $[0,1]$, we can check that the map $x \to Z_\infty(x)$ still provides a transport between μ and ν

2 Gravitational allocation scheme

Let

$$f(z) := \sum_k \xi_k \frac{z^k}{\sqrt{k!}} \tag{2.1}$$

be the Gaussian entire function with first intensity measure equal to π^{-1}Leb. Here the ξ_k are independent standard complex Gaussians [2]. Let $\mathcal{Z}_f$ denote the random zero set of f, and

$$n_f := \sum_{x \in \mathcal{Z}_f} \delta_x$$

be the associated atomic measure. We know from the previous sessions that the set $\mathcal{Z}_f$ is translation invariant in law (and that this in fact characterises the analytic function f up to a one-parameter family).

The main object of today's session is the paper of Nazarov, Sodin and Volberg [NSV07], which constructs an explicit transport between Lebesgue measure $m(dx)$ on $\mathbb{C}$, and the random measure πn_f. That is, the authors define an explicit map $T : \mathbb{C} \to \mathbb{C}$ (that is random but a

[2] with density $\pi^{-1}\, \mathrm{e}^{-|z|^2}$ with respect to Lebesgue measure

deterministic function of f) such that almost surely n_f is the image measure of m under T. Note that this differs from the framework of Section 1, since both m and n_f have infinite mass, but it turns out that a transport can be constructed in a very similar way. We interpret such a transport as a way to fairly allocate regions of the complex plane to each of the zeroes of f; indeed, if we have such a map then by definition, the Lebesgue mass of $\{y : T(y) = x\}$ has to be equal to π for each x in the zero set of f. The transport map in this case is often referred to as an *allocation*.

Remark 2.1 *Note that the first intensity measure of πn_f is equal to Lebesgue measure. For another Gaussian entire function g with translation invariant zero set (which must have first intensity equal to some other multiple of Lebesgue measure) an allocation can only be defined from Lebesgue to a different multiple of n_g. This can be done easily by modifying the construction below, so from now on we will stick to the case when f is given by (2.1).*

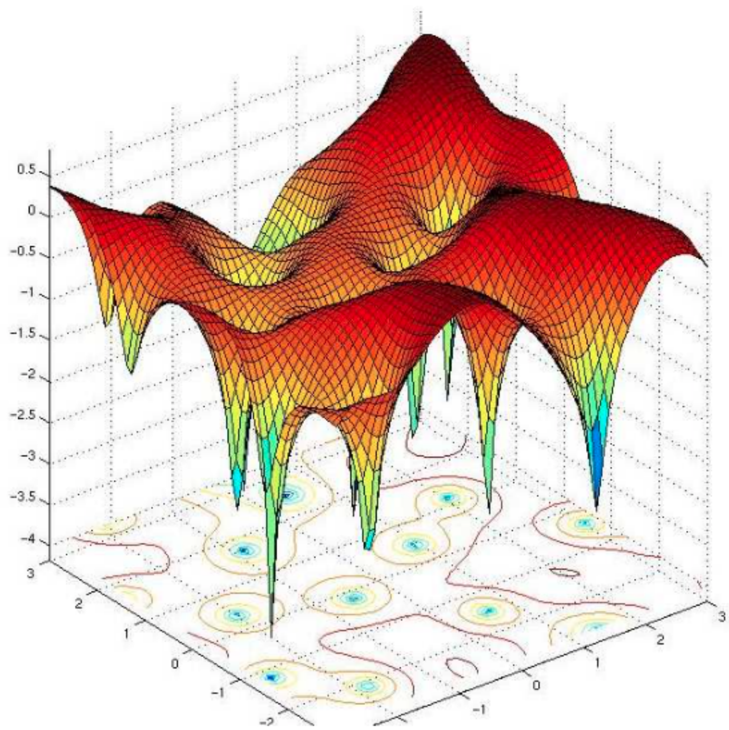

Figure 1: The function $u(z) = \log|f(z)| - \frac{1}{2}|z|^2$ (from [NSV07])

The transport is constructed as follows. Let

$$u(z) = \log|f(z)| - \frac{1}{2}|z|^2 \tag{2.2}$$

and define, for $z \in \mathbb{C} \setminus \mathcal{Z}_f$, $X_t(z)$ to be the solution of

$$\frac{dX_t(z)}{dt} := -\nabla u(z) \;\; ; \;\; X_0(z) = z \tag{2.3}$$

on the maximal interval $[t_-(z), t_+(z)] \subset \mathbb{R}$ on which it is defined (we will see later, in Section 4, that there is a unique maximal solution with $t_-(z) \in [-\infty, 0)$ and $t_+(z) \in (0, \infty]$). Let Γ_z denote the image $\{X_t(z) : t \in [t_-(z), t_+(z)]\}$, which we think of as an oriented curve, up to time reparameterisation [3]. For $a \in \mathcal{Z}_f$ define

$$B(a) := \{z \in \mathbb{C} : X_{t_+(z)}(z) = a\} \cup \{a\} \tag{2.4}$$

Thus the *basin* $B(a)$ is the collection points that are transported to a under the flow (2.3). We call the curves Γ_z *gradient curves.*

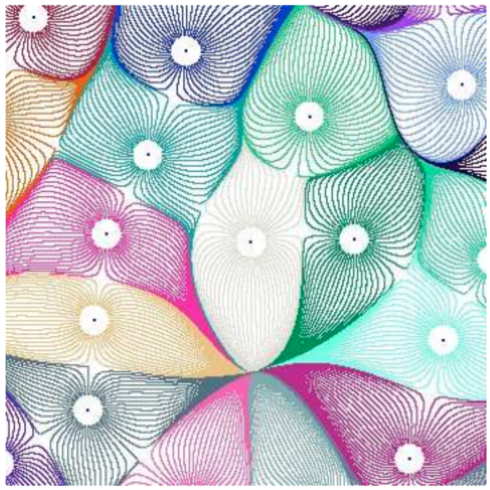

Figure 2: The basins $\{B(a) : a \in Z_f\}$ (from [NSV07]).

Theorem 2.2 ([NSV07]) *Almost surely, each basin $B(a)$ is bounded by finitely many smooth gradient curves, has area equal to π, and $\mathbb{C} = \cup_{a \in \mathcal{Z}_f} B(a)$ up to a set of zero-Lebesgue measure.*

This means that, almost surely, the map sending each $z \in \mathbb{C}$ to the basin that contains it is an allocation of Lebesgue measure to πn_f. Recall from the previous sessions that

$$\Delta u = 2\pi n_f - 2m \tag{2.5}$$

in the distributional sense, and hence, this allocation is very close to that described in Section 1.2.

Remark 2.3 *Note that this allocation does depend on the analytic function f, and not just on the set of zeroes. Indeed, we will see that ∇u has vanishing normal derivative on the boundaries of the basins, and it is easy to see that if one replaces f with $e^g f$ for some analytic g, then this will not be preserved.*

Before stating some more results from [NSV07] concerning this scheme, let us first see why the area of each basin has to be exactly equal to π.

Proof of equal area basins, assuming they are bounded by finitely many smooth gradient curves. The idea behind the proof is to use that fact that $\Delta u = 2\pi n_f - 2\pi m$ (in the distributional sense)

[3] so we can and will have $\Gamma_z = \Gamma_w$ for $z \neq w$

and then to use Green's theorem and the fact that our boundary is made up of smooth gradient curves.

To do this rigorously, take $a \in Z_f$ and first observe that if we write

$$\begin{aligned} u(z) &= u_1(z) + u_2(z) \\ &:= \log|z-a| + (u(z) - \log|z-a|) \end{aligned}$$

then u_2 is analytic in $B(a)$ with $\Delta u_2 = -2$ in $B(a)$, and u_1 is analytic in $B(a)\setminus\{a\}$ with $\Delta u_1 = 0$ in $B(a)\setminus\{a\}$. Set $D_\varepsilon = B(a)\setminus D(a,\varepsilon)$. Then we have

$$\int_{D_\varepsilon} \Delta u(z) dz = -2m(D_\varepsilon) \overset{\varepsilon\to 0}{\to} m(B(a)) \tag{2.6}$$

almost surely as $\varepsilon \to 0$. Moreover, we can write

$$\int_{D_\varepsilon} \Delta u(z) dz = \int_{\partial B(a)} \partial_n u(z)\, dz - \int_{\partial D(a,\varepsilon)} \partial_n u(z) \tag{2.7}$$

by the Green formula, since we are assuming that the boundary of $B(a)$ is given by finitely many smooth curves (where the normal derivative means the scalar product of ∇u with the outward pointing normal to the boundary). We claim that the first term is equal to 0, and that $\int_{\partial D(a,\varepsilon)} \partial_n u(z) \to 0$ as $\varepsilon \to 0$. This completes the proof by combining (2.6) and (2.7).

So let us prove the claim. For the second statement, we write

$$\lim_{\varepsilon\to 0} \int_{\partial D(a,\varepsilon)} \partial_n u(z) = \lim_{\varepsilon\to 0} \int_{\partial D(a,\varepsilon)} \partial_n u_1(z)$$

which follows because $\partial_n u_2$ is smooth and so bounded on $\partial D(0,\varepsilon)$, and we are integrating over a curve of length going to 0. Then because $u_1(z) = \log|z-a|$ we can apply Cauchy's theorem to see that the right hand side of the above is equal to 2π. For the first statement we use the assumption that $\partial B(a)$ is made up of gradient curves of u. Suppose for contradiction that there is some $z_0 \in \partial B(a)$ such that $\nabla u(z_0)$ has non-zero component in the direction of the normal to $\partial B(a)$. This implies that Γ_{z_0} crosses [4] the boundary of $\partial B(a)$ at z_0. On the other hand, since $\partial B(a)$ is made up of gradient curves and there is a.s. a unique gradient curve passing through z_0 (see Section 4), Γ_{z_0} should be a subset of $\partial B(a)$. This is a contradiction.

□

3 Results and interpretation

Having constructed an allocation of Lebesgue measure to πn_f, the authors in [NSV07] then ask about the *cost-effectiveness* of the model. Of course for this mass transport the total cost as defined by (1.1) must be infinite, but we can ask for example, for information on the tails of $|T(z) - z|$ for a given point z. The better T is localised, the more uniformly spread we expect the zeroes of f to be.

First, we observe that any given $z \in \mathbb{C}$ is almost surely contained in some basin. This follows from Theorem 2.2 and the following lemma (which will also be important in a lot of what follows):

[4] i.e. at some non-zero angle

Lemma 3.1 *For any $z_0 \in \mathbb{C}$,*

$$u(z_0 + z) \stackrel{(d)}{=} u(z).$$

Proof. Define

$$T_{z_0} f(z) := f(z_0 + z)\,\mathrm{e}^{-z\bar{z_0}}\,\mathrm{e}^{-\frac{1}{2}|z_0|^2}.$$

Then a direct calculation (for the covariances of $T_{z_0} f(z)$) yields that $T_{z_0} f \stackrel{(d)}{=} f$. Moreover, we have that

$$\begin{aligned} u(z_0 + z) &= \log|f(z_0 + z)| - \frac{1}{2}|z_0 + z|^2 \\ &= \log|f(z_0 + z)| - \Re(z\bar{z_0}) - \frac{1}{2}|z|^2 - \frac{1}{2}|z_0|^2 \\ &= \log|T_{z_0} f(z)| - \frac{1}{2}|z|^2, \end{aligned}$$

where by the invariance of $T_{z_0} f(z)$ in law, the last expression has the same law as $u(z)$. □

Now for fixed z we denote by B_z the basin that contains the point z (which is almost surely well defined by the above comments). In [NSV07], the following theorem is proven concerning the asymptotics of the size of B_z.

Theorem 3.2 *For any $z \in \mathbb{C}$ and $R \geq 1$,*

$$ce^{-CR(\log R)^{3/2}} \leq \mathbb{P}(diam(B_z) > R) \leq Ce^{-cR(\log R)^{3/2}} \tag{3.1}$$

where $C > 0$ and $c > 0$ are absolute numerical constants.

Let us compare this quickly with another result from the literature. In the paper [ST06] it is shown that a random set of points in $\mathbb{C}$ with the same law as $\mathcal{Z}_f$ can be constructed by taking a peturbation of the square lattice:

$$\mathcal{Z}_f \stackrel{(d)}{=} \{\sqrt{\pi}(k + \mathrm{i}\,l) + \xi_{k,l} : (k, l) \in \mathbb{Z}^2\}$$

where the $\xi_{k,l}$ have subgaussian tails. This clearly provides another way to allocate Lebesgue measure to πn_f (although this is non-explicit, since the construction in [ST06] is non-explicit). It is claimed in [NSV07] that a modification of this scheme yields a transport ,map T from m to πn_f such that the tail probabilities $\mathbb{P}(|T(z) - z| > R)$ decay like $\exp(-R^4(\log R)^{-1})$ rather than $\exp(-R(\log R)^{3/2})$. The reason that the tail probabilities are higher for the gravitational allocation scheme is the existence of long, thin "tentacles" around some basins. This is made precise in [NSV07] through the following result:

Theorem 3.3 *For any $z \in \mathbb{C}$, $\epsilon > 0$ and $R \geq 1$*

$$c(\varepsilon)\,\mathrm{e}^{-C(\varepsilon)R^4} \leq \mathbb{P}(m(B_z \setminus D(a_z, R)) \geq \varepsilon) \leq C(\varepsilon)\,\mathrm{e}^{-c(\varepsilon)R^4}$$

where $c(\varepsilon)$ and $C(\varepsilon)$ are positive constants depending on ε.

This says that we can throw away, say, one percent of the area of the basin B_z containing z, and then the probability that B_z has diameter greater than R will decay like e^{-R^4}. The authors in [NSV07] also provide a combinatorial procedure to modify the basins (in a "small way"), such that the probability of any given modified basin having diameter larger than R decays like $\mathrm{e}^{-R^4(\log R)^{-3/2}}$.

In fact, we cannot hope for a much faster rate of decay than this since it is proved in [ST05] that the *hole probability* $\mathbb{P}(\mathcal{Z}_f \cap R\mathbb{U} = \emptyset)$ is bounded below by $c\,\mathrm{e}^{-CR^4}$.

Remark 3.4 *The gravitational allocation scheme has also been considered as a method for allocating Lebesgue measure on $\mathbb{R}^d$ to the points Π of a Poisson process with intensity 1 on $\mathbb{R}^d$. In contrast to the above, the allocation scheme is only shown to work (i.e. to yield a transport map) when $d \geq 3$. One key difference is that in our case, we define $u(z) = \log|f(z)| - \frac{1}{2}|z|^2$ and let points flow according to the vector field*

$$\nabla u(z) = g(z) - z + \sum_{a \in \mathcal{Z}_f} \frac{(z-a)}{|z-a|^2},$$

(for some analytic g). Here the sum is almost surely convergent away from $\mathcal{Z}_f$ by analyticity of f.

For the Poisson point process Π, one would analogously like points to flow according to the vector field

$$F(z) = z - \sum_{a \in \Pi} \frac{(z-a)}{|z-a|^2}$$

but in this case, it is not clear that the sum will converge. When $d \geq 3$ however, the natural analogue of the gravitational allocation scheme yields a vector field F, where the exponent 2 in the denominator above is replaced with d. In this case, the sum is known to converge almost everywhere almost surely.

4 Preliminaries on gradient curves

In this section we recall some basic properties of gradient curves that we will need in what follows. The material comes mainly from [Her66].

We are interested in solutions of the following equations:

$$\frac{dX_t(z)}{dt} := -\nabla u(z) \;\; ; \;\; X_0(z) = z \tag{4.1}$$

where z is any point in $D := \mathbb{C} \setminus \mathcal{Z}_f$ and $u(z) := \log|f(z)| - \frac{1}{2}|z|^2$ is as defined in Section 2. Note that ∇u is analytic on D. By standard ODE theory it then follows that for every z in D, there exists a unique solution to (4.1) defined on $(t_-(z), t_+(z))$ with $t_\pm(z) \in (0, \pm\infty]$. Moreover, if $t_-(z) > -\infty$ or $t_+(z) < \infty$ (or both) we have $X_t(z) \to \partial D$ (meaning that X_t approaches a point of $\mathcal{Z}_f$) or $X_t(z) \to \infty$ as $t \to t_\pm(z)$. We write Γ_z for the curve $\{X_t(z) : t \in [t_-(z), t_+(z)]\}$ and call this a *gradient curve*. We think of the curve as an oriented curve, but up to time reparameterisation (so we can have $\Gamma_z = \Gamma_w$ for $z \neq w$). By uniqueness of solutions to (4.1) there is exactly one gradient curve passing through each $z \in D$.

We call points $w \in D$ with $\nabla u = 0$ *singularities* of (4.1) and denote the set of all such points by $\mathrm{Crit}(u)$. $w \in \mathrm{Crit}(u)$ are exactly the points such that $X_t(w) = w$ for all $t \in (-\infty, \infty)$ and are clearly the only points (other than ∞) that can be approached by any $X_t(z)$ as $t \to -\infty$ or $t \to \infty$.

Hence, any oriented gradient curve Γ has a *starting point* $s(\Gamma) \in \mathrm{Crit}(u) \cup \{\infty\}$ and a terminating point $t(\Gamma) \in \mathrm{Crit}(u) \cup \mathcal{Z}_f \cup \{\infty\}$. We have already ruled out the possibility that an oriented curve Γ has $s(\Gamma) \in \mathcal{Z}_f$ because $-\nabla u(z) \to +\infty$ as $z \to a \in \mathcal{Z}_f$ (in particular, for any $a \in \mathcal{Z}_f$ there exists some $\rho > 0$ such that $-\nabla u$ points outwards from $\partial D(a,\rho)$ at every point of $\partial D(a,\rho)$).

4.1 Classification of singularities

A singularity $w \in \mathrm{Crit}(u)$ is called *simple* if

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \neq 0 \tag{4.2}$$

Lemma 4.1 *Simple singularities are isolated.*

Proof. By the inverse function theorem, if w is a simple singularity in $\mathrm{Crit}(u)$, then ∇u is invertible in a neighbourhood of w. This implies that $\nabla u \neq 0$ in a neighbourhood of w, and hence w is an isolated singularity. □

Moreover, at a simple singularity $w \in \mathrm{Crit}(u)$, one can determine the behaviour of gradient curves approaching w by approximating solutions to (4.1) with a linearisation of the system. It then follows, [Her66, Chapter 4], that the behaviour can be classified according to the eigenvalues of the matrix

$$\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix}$$

evaluated at w. Note that the eigenvalues of this matrix must both be real, and so following [Her66] we see that we have three possibilities:

(i) both eigenvalues are positive. Gradient curves can terminate, but not start, at such a point.

(ii) both eigenvalues are negative. Gradient curves can start, but not terminate, at such a point.

(iii) one eigenvalue is positive, and one is negative. There are at most two gradient curves starting and two gradient curves terminating at such a point (in addition to the constant gradient curve that stays at the point).

This fits well with our intuition (given the definition of (4.1)) since points satisfying (i) are local minima of u, points satisfying (ii) are local maxima, and points satisfying (iii) are saddle points. We also observe here that since $\nabla u \leq 0$ on D, u in fact has no local minima on D.

Finally we remark that since ∇u is analytic on D, any gradient curve is smooth away from its starting and terminating points.

5 Proofs

The key result from which Theorems 2.2 and 3.2 follow is the following theorem concerning the existence of long gradient curves (which will be discussed in next week's session). For $w \in \mathbb{C}$ and $s \geq 0$, we let $Q(w, s)$ be the square of side length $2s$ centered at w.

Theorem 5.1 ([NSV07]) *There exist absolute constants $c, C > 0$ such that for any $R \geq 1$,*

$$\mathbb{P}\big(\textit{there exists a gradient curve joining } \partial Q(0,R) \textit{ with } \partial Q(0,2R)\big) \leq C\,\mathrm{e}^{-cR(\log R)^{3/2}}.$$

Let us now see how to prove Theorem 2.2 assuming that the above is true. We first need the following lemma.

Lemma 5.2 *Almost surely, every $w \in Crit(u)$ is simple.*

Proof. We will begin by deriving conditions for $w \in \mathbb{C}$ to be a non-simple critical point. Then we will show that for these conditions to be satisfied anywhere, we are putting too many constraints on the independent ξ_k's. This will imply that, almost surely, no such points exist.

Firstly, for w to be a non-simple singularity, we have the obvious requirement that $\nabla u(w) = 0$ (i.e. w is *a* singularity). Note that $w = 0$ is almost surely not in $\mathrm{Crit}(u)$ (by translation invariance of u) and so from now on we may assume that $w \neq 0$. We write $u = u_1 + u_2$ where $u_1(z) = \log|f(z)|$ and $u_2(z) = -|z|^2/2$. Then on D, u_1 is the real part of the analytic function $F(z) = \log f(z)$, and so

$$\nabla u_1(z) = \overline{F'(z)} = \overline{\frac{f'(z)}{f(z)}}.$$

We can also calculate that $\nabla u_2 = -z$ and hence

$$w \in \mathrm{Crit}(u) \Rightarrow \overline{\frac{f'(w)}{f(w)}} - w = 0 \Leftrightarrow f(w) = \bar{w}^{-1} f'(w). \tag{5.1}$$

Similarly, we can calculate that

$$\frac{\partial^2 u_1}{\partial x^2}\frac{\partial^2 u_1}{\partial y^2} - \left(\frac{\partial^2 u_1}{\partial x \partial y}\right)^2 = -\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 - \left(\frac{\partial^2 u_1}{\partial x \partial y}\right)^2 = -|F''|^2 = -\left|\left(\frac{f'}{f}\right)'\right|$$

where we have used the Cauchy–Riemann equations for the first equality.

Since $\frac{\partial^2 u_2}{\partial x^2}\frac{\partial^2 u_2}{\partial y^2} - \left(\frac{\partial^2 u_2}{\partial x \partial y}\right)^2 = 1$ we see that for $w \in \mathrm{Crit}(u)$ to be non-simple, we also require that

$$\left|\bar{w}\frac{f''}{f'}(w) - \bar{w}^2\right|^2 = 1 \tag{5.2}$$

where we have replaced $|(\frac{f'}{f})'(w)|$ with $|\bar{w}\frac{f''}{f'}(w) - \bar{w}^2|^2$ by (5.1).

Now we will show that (5.1) plus (5.2) is too much to ask for. Indeed, if

$$f(z) = \xi_0 + \xi_1 z + \sum_{k \geq 2} \xi_k \frac{z^k}{\sqrt{k!}} =: \xi_0 + \xi_1 z + h(z),$$

then conditionally on $h(w)$ (equivalently on $\{\xi_k\}_{k\geq 2}$), for (5.2) to be satisfied at w, it must hold that

$$\left|\bar{w}\frac{h''(w)}{h'(w) + \xi_1} - \bar{w}^2\right| = 1.$$

Since ξ_1 is independent of $\{\xi_k\}_{k\geq 2}$ and Gaussian, the conditional probability that the above holds at a fixed w, given $\{\xi_k\}_{k\geq 2}$, is equal to 0. Thus by Fubini, the set of w for which (5.2) holds is a.s. a set of measure 0, and is determined entirely by $\{\xi_k\}_{k\geq 1}$. Let us call this set A.

We also write A' for the set of $w \in \mathbb{C}$ at which (5.1) holds. We conclude by showing that $\mathbb{P}(A' \cap A \neq \emptyset) = 0$. This implies that with probability 1, no $w \in \mathbb{C}$ satisfies both (5.1) and (5.2) simultaneously, and hence completes the proof.

To see why this final claim is true, write

$$\begin{aligned}\mathbb{P}(\exists w \in A \text{ s.t. } f(w) = \bar{w}^{-1} f'(w)) &= \mathbb{E}\left(\mathbb{P}(\exists w \in A \text{ s.t. } f(w) = \bar{w}^{-1} f'(w) \,|\, \{\xi_k\}_{k\geq 1})\right) \\ &= \mathbb{E}\left(\mathbb{P}\big(\exists w \in A : \xi_0 = \frac{g'(w)}{\bar{w}} - g(w)\big)\right) \\ &= \mathbb{E}\left(\mathbb{P}(\xi_0 \in \tilde{A})\right),\end{aligned}$$

where $\tilde{A}$ is a random set that has Lebesgue measure 0 almost surely and is independent of ξ_0 (here $g(z) := \xi_1 z + h(z)$ is measurable with respect to $\{\xi_k\}_{k\geq 1}$.) Since ξ_0 is Gaussian, this is indeed equal to 0. □

Proof of Theorem 2.2. By the arguments given in Section 2, we need only prove that, almost surely, $\mathbb{C} = \cup_{a\in\mathcal{Z}_f} B(a)$ up to a set of Lebesgue measure 0, and that every basin $B(a)$ is bounded by finitely many smooth gradient curves.

The long gradient curve theorem, Theorem 5.1, immediately tells us that there are a.s. no gradient curves starting or terminating at ∞. This, together with the considerations from Section 4 and Lemma 5.2, means that all the points in $\mathrm{Crit}(u)$ are isolated, and that every gradient curve has a starting point which is an local maximum of u or a saddle point of u, and a terminating point which is either in $\mathcal{Z}_f$ or is a saddle point of u. The gradient curves are also smooth away from their endpoints, and the diameters of all the basins are finite, since otherwise they would contain a gradient curve of infinite length.

Let S be the set of all gradient curves Γ that terminate at a saddle point of u (rather than a zero of f). Note that any point of $\mathbb{C}$ that is not in some basin $B(a)$ with $a \in \mathcal{Z}_f$ must lie on a gradient curve $\Gamma \in S$. We claim that almost surely, every bounded $K \subset \mathbb{C}$ intersects only finitely many $\Gamma \in S$. To see this, first observe that almost surely, every such K must be contained in a compact set $\tilde{K}$ that itself contains all the gradient curves $\Gamma \in S$ such that $\Gamma \cap K \neq \emptyset$. Indeed, if this were not the case then there would exist some $N \in \mathbb{N}$ such that $\partial Q(0, N)$ is joined to $\partial Q(0, M)$ by a gradient curve for every $M > N$, and for fixed N this has zero probability by Theorem 5.1. Now, for every K, the corresponding compact set $\tilde{K}$ must contain all of the points $t(\Gamma)$ with $\Gamma \in S$ and $\Gamma \cap K \neq \emptyset$, and since these are all isolated and $\tilde{K}$ is compact, the set $\{t(\Gamma) : \Gamma \in S \text{ and } \Gamma \cap K \neq \emptyset\}$ must be finite. Since any saddle point of u serves as a terminating point $t(\Gamma)$ for at most 2 distinct (non-trivial) gradient curves Γ, this implies that the set of gradient curves in S intersecting K must be finite.

In particular, we see that for any $N \in \mathbb{N}$, the Lebesgue measure of $\{z \in N\mathbb{U} \setminus \cup_{a\in\mathcal{Z}_f} B(a)\}$ is equal to zero with probability one. Hence, almost surely we have $\mathbb{C} = \cup_{a\in\mathcal{Z}_f} B(a)$ up to a set of Lebesgue measure zero. Moreover, since every $B(a)$ has finite diameter, it must be bounded by finitely many gradient curves in S, which we already know are smooth. This completes the proof.

□

Finally, what is the idea to prove Theorem 3.2?

Firstly, by the translation invariance of u (Lemma 3.1) we see that

$$\operatorname{diam}(B_z) \overset{(d)}{=} \operatorname{diam}(B_w) \tag{5.3}$$

for any $z, w \in \mathbb{C}$. Therefore, by considering B_0 and applying Theorem 5.1, we obtain the upper bound in Theorem 3.2 immediately (indeed, if $\operatorname{diam}(B_0) > R$ then either $B_0 = B(a)$ with $|a| > R/3$, so there exists a gradient curve from $\partial Q(0, R/20)$ to $\partial Q(0, R/10)$, or $|a| < R/2$ and there exists a gradient curve from $\partial Q(0, R/3)$ to $\partial Q(0, 2R/3)$).

For the lower bound, the argument is more complicated, and we just give a very brief idea here. See [NSV07, Section 9] for the detailed proof. The idea is to just consider R such that $R = \sqrt{n}$ for $n \in \mathbb{N}$, and to show that for any such R, the point iR is contained in a gradient curve of length $> c'R$ (some absolute $c' > 0$) with probability bounded below by $c\,\mathrm{e}^{-CR(\log R)^{3/2}}$. Then by (5.3) the lower bound follows.

To see why the point $iR = i\sqrt{n}$ has some reasonable chance to be contained in a long gradient curve, imagine replacing the Gaussian entire function $f(z)$ with the function $g(z) := \frac{z^n}{\sqrt{n!}}$. If we set $u_g(z) := \log|g(z)| - |z|^2/2$, then we see that $\nabla u_g(z) = 0$ on the circle $\{|z| = n\}$. It is then not too hard to see that if one replaces g with, say,

$$F(z) := \frac{z^n}{\sqrt{n!}}\left(1 + \frac{z}{10R}\right)$$

(this is the choice in [NSV07]) then near $\{|z| = \sqrt{n}\}$, if $u_F(z) := \log|F(z)| - |z|^2/2$, $-\nabla u_F(z)$ will still have a small component, $O(R^{-1})$, in the direction of z, and an angular component of comparable size in the anti-clockwise direction. This means that if we instead considered the gradient curves of u_F, the point iR would be contained in a gradient curve of length $O(R)$.

In fact, the same will be for the gradient curves of u (the ones that we are interested in), as long as f is not too big a peturbation of F in the annulus $\{R - 2 \le |z| \le R + 2\}$, say. More precisely, in [NSV07] it is shown that on the event that $|(f - F)/F| \le R^{-2}$ in this annulus, iR will be contained in a gradient curve of length $\ge c'R$ for some absolute constant $c' > 0$.

The rest of the proof consists of estimating from below the probability of this event, and we direct the reader to [NSV07] for these arguments.

6 Questions

- Can we use properties of the gravitational allocation to obtain any of the results that we have already seen for zeroes of f? For example, the probability of having many zeroes in a disc, or the hole probability, should be closely related to the size of the basins.

- If we consider the Gaussian analytic function

$$f(z) := \sum_{n=0}^{L} \xi_n \frac{\sqrt{L(L-1)...(L-n+1)}}{\sqrt{n!}} z^n$$

  for some $L \in \mathbb{N}$ (this is a Gaussian analytic function whose zeroes are invariant under isometries of $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$), what can we say about the gravitation allocation scheme? In two weeks time, Nina will talk about gravitational allocation for uniformly distributed points on the sphere, [HPZ17].

- Going back to the original Gaussian entire function f; does there exist a transport T of m to πn_f such that $\sup_{z\in\mathbb{C}} \mathbb{P}(|T(z)-z|>R)$ decays as e^{-R^4}?
- How long does it typically take for the gradient curve Γ_z of a given $z \in \mathbb{C}$ to reach its terminal point in $\mathcal{Z}_f$?
- Can we say anything more about the *optimality* of this transport? Of course, the total cost of the transport given by gravitational allocation will be infinite, but we can ask for some renormalised cost: for example, the limit of total cost on a large disk, renormalised by its area. It has been shown in [HS13] that for allocation of Lebesgue measure to points of a Poisson process with unit intensity, there exists a unique scheme minimising the expectation of this cost whenever $d \geq 3$. Is the same result true for zeroes of the Gaussian entire function? If so, how close is the gravitational allocation scheme to being optimal?

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