Gravitational Allocation II - Absence of long gradient curves

0. Introduction

Recall from last week:

$$f(z) = \sum_{k\geq 0} \xi_k \frac{z^k}{\sqrt{k!}}, \quad \xi_k \sim N_{\mathbb{C}}(0,1)$$

the Gaussian entire function (G.E.F.) with first intensity measure $\frac{1}{\pi} m_2$.

We associated to f a random potential

$$u(z) = \log|f(z)| - \frac{1}{2}|z|^2$$

(well defined on $\mathbb{C} \setminus \underbrace{Z_f}_{\text{set of zeroes of } f}$) and constructed an allocation

$$T: \mathbb{C} \longrightarrow \mathbb{C} \quad \text{s.t.} \quad m_2 \longrightarrow \pi n_f, \quad n_f = \sum_{a\in Z_f} \delta_a$$

(by defining $B(a) = \{z\in\mathbb{C} : X_{t_+(z)}(z) = a\} \cup \{a\}$ and $T(z) := a$ for $z \in \underbrace{B(a)}_{\text{basin}}$).

For this, we needed

Theorem (Nazarov, Sodin, Volberg '05)

$Q(w,s)$ = square in $\mathbb{C}$ with center $w\in\mathbb{C}$, side-length $s>0$, $\partial Q(w,s)$ its boundary;

$$\mathbb{P}(\exists \underbrace{\text{gradient curve}}_{\text{solution to } \frac{d}{dt}z(t) = \nabla u(z(t))} \partial Q(0,R) \longleftrightarrow \partial Q(0,2R)) \leq Ce^{-cR\sqrt{\log R}}$$

for $R >> 1$.

[Remark: Holds also with $Ce^{-cR(\log R)^{\frac{3}{2}}}$ as an upper bound, but the proof is more involved.]

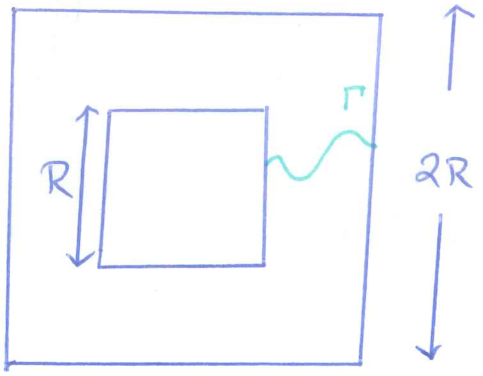
Aim of today's session: Prove this Thm. (at least partially).

1. Outline of the proof

1.) Recall: u is shift-invariant in law (unlike f)

- Pretend that u & its derivatives up to second order are bounded (*) $\longleftarrow$ not true!

Then, if a $\underbrace{\text{long gradient curve}}_{\text{connecting } \partial Q(0,R),\ \partial Q(0,2R)}$ Γ exists, $|\nabla u|$ must be "small" on Γ: we have $|\nabla u| \leq \frac{C}{R}$ on Γ.



Since $\int_\Gamma |\nabla u||dz| = \int |\nabla u(z(t))|^2 dt$

$= \max_{z_1, z_2 \in \Gamma} |u(z_2) - u(z_1)|$

$\leq C$

and $\int_\Gamma |dz| = \text{Length}(\Gamma) \geq \frac{1}{2}R$

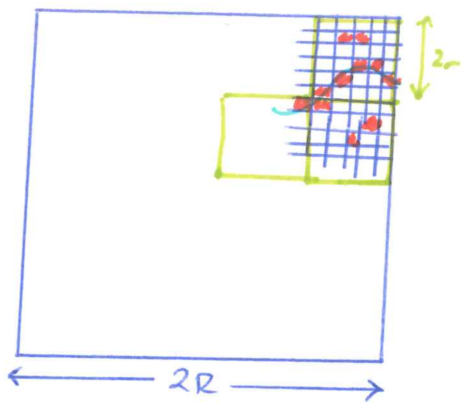
(if $|\nabla u| > \frac{2}{R}C$ on Γ, $\int_\Gamma |\nabla u||dz| > \frac{2}{R} \cdot \frac{1}{2}R \cdot C = C$)

2.) Discretization of the problem: Consider squares $Q(w, \frac{1}{R})$

If $|\nabla u| \leq \frac{C}{R}$ on one point $Q(w, \frac{1}{R}) \underset{(\partial_{ij} u)_{ij} \text{ bounded}}{\Longrightarrow} |\nabla u| \leq \frac{C'}{R}$ on $Q(w, \frac{1}{R})$ and especially

$$|\nabla u(w)| \leq \frac{C'}{R}.$$

We call these squares red.



Idea: A long gradient curve Γ leaves a trace of red squares.

How many red squares are there? N.

- $\left(2R/(\frac{1}{R})\right)^2 = 4R^4$ squares in total, $Q(w, \frac{1}{R})$

- $P(|\nabla u(w)| \leq \frac{C}{R}) \underset{\text{shift-inv.}}{=} P(|\nabla u(0)| \leq \frac{C}{R})$

$\underset{\nabla u = \frac{f'(z)}{f(z)} - z}{=} P\left(\left|\frac{\xi_1}{\xi_0}\right| \leq \frac{C}{R}\right) \leq \frac{C}{R^2}$

$$\Longrightarrow E[N] \leq CR^{-2} \cdot R^4 = CR^2$$

⟶ this would "barely be enough" to make a connection between $\partial Q(0,R)$ & $\partial Q(0,2R)$)

⟶ the red squares have to be "aligned" in one direction, this "seems unlikely".

Can we quantify this?

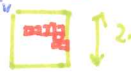
3.) Introduce an intermediate scale $\frac{1}{R} < r < R$:

If we consider $Q(w, 2r)$, then

$$E[\underbrace{N_r}_{\substack{\text{red squares}\\ \text{in } Q(w,2r)}}] = \underbrace{CR^{-2}}_{\substack{\text{prob. of}\\ \text{being red}}} \cdot \left(\frac{2r}{(\frac{1}{R})}\right)^2 = Cr^2$$

$$\Longrightarrow P(N_r \geq rR) \overset{\text{Markov}}{\leq} \frac{Cr^2}{rR} = C\frac{r}{R} \xrightarrow[R\to\infty]{} 0 \quad \text{if } r(R) \text{ grows less than linear.}$$

To connect $\partial Q(w,r)$ to $\partial Q(w,2r)$, we need $\sim rR$ red squares!

⟶ it is unlikely to have a noticable presence of red squares of side-length $\frac{1}{R}$ within a block of side-length $2r$. $P($"  "$) \leq C\frac{r}{R}$.

4.) Almost independence: Neighboring blocks are (of course) correlated, but what happens with "distant" blocks?

Since $E[f(w')\overline{f(w'')}] = e^{w'\overline{w''}}$

$$\Longrightarrow f(w')e^{-\frac{|w'|^2}{2}},\ f(w'')e^{-\frac{|w''|^2}{2}} \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

with covariance $e^{-\frac{1}{2}|w'-w''|^2}$

⤳ Require $e^{-\frac{1}{2}|w'-w''|^2} \overset{!}{<<} \frac{1}{R} \Longrightarrow |w'-w''| > A\sqrt{\log R}$

$A >> 1$. Then blocks at distance $> A\sqrt{\log R}$ are morally independent. (✶✶)

↳ "Toy model": $Q(0, 2R)$ divided into squares of size $\frac{1}{R}$, $P(\text{"red"}) = \frac{1}{R^2}$ for a single square, and indep. if distance of their centers is $\geq r = \sqrt{\log R}$.

↳ look at blocks of size $2r$, roughly speaking, all of these are independent.

A chain of red squares identifies a family of neighboring blocks (size r) ~~s.t.~~ with a noticable red path inside $Q(w,2r)$.
$Q(w,r)$

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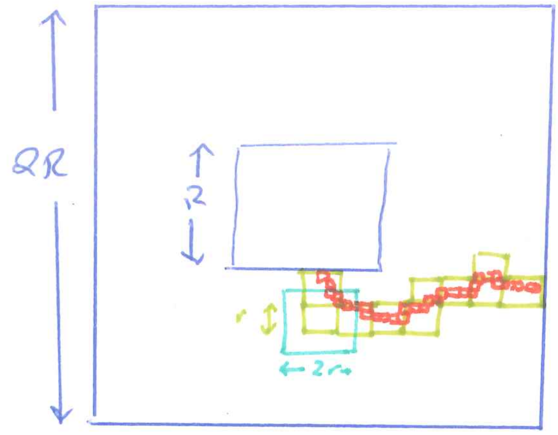


→ a priori estimate that this happens?

- Prob. that a chain of L blocks all have significant red part:

$$\sim \left(\frac{r}{R}\right)^L \le e^{-cL\log R}$$

↑ independence

- $\sim \left(\frac{R}{r}\right)^2$ possibilities to "start" a chain
- at each step: 4 possibilities to move

$$\rightsquigarrow \left(\frac{R}{r}\right)^2 e^{CL} \text{ "chains" of blocks.}$$

$$\Longrightarrow P(\exists \text{ long gradient curve})$$

$$\le \sum_{L \ge \frac{R}{2\sqrt{\log R}}} e^{-cL\log R} \cdot \left(\frac{R}{r}\right)^2 e^{+CL}$$

$$\overset{R\gg 1}{\le} \sum_{L \ge 2\frac{R}{\sqrt{\log R}}} e^{-c'L\log R} \le Ce^{-c'L\log R}.$$

2. Almost independence

We want to make (**) more precise:

Proposition 1

$\forall M > 0 \ \exists A = A(M) > 1$: $\forall r > 1$ and all families of points $(w_j)_j \subseteq \mathbb{C}$: $|w_i - w_j| \ge Ar$:

$$\underbrace{T_{w_j} f}_{T_w f = f(w+z)e^{-z\bar{w}}e^{-\frac{1}{2}|w|^2}} = f_j + h_j$$

where f_j are independent G.E.F. and h_j are random analytic functions with

$$P\left(\max_{|z|\le r} |h_j(z)| > e^{-Mr^2}\right) \le 2e^{-\frac{1}{2}Mr^2}$$

Ingredients of proof:

$$T_w f(z) = \sum_{u\geq 0} \frac{\xi_u(w)}{\sqrt{u!}} z^u,$$

- bound $\left|\mathbb{E}[\xi_j(w')\overline{\xi_u(w'')}]\right|$ and thus $\sum_{(u,j)\neq(l,i)} \left|\mathbb{E}[\xi_u(w_j)\overline{\xi_l(w_i)}]\right|$
- use Lemma A2: $\xi_u(w_j) = \overbrace{\zeta_u(w_j)}^{\text{indep. } \forall u,j} + \overbrace{b_{uj}\eta_u(w_j)}^{\text{exp. small}}$
- $f_j(z) = \sum_{u\geq 0} \zeta_u(w_j)\frac{z^u}{\sqrt{u!}}$ are indep. G.E.F.
- $h_j(z) = T_{w_j}f(z) - f_j(z)$, use Lemma A1. ($\square$)

3. Almost boundedness of u

Here, we make $(*)$ more precise:

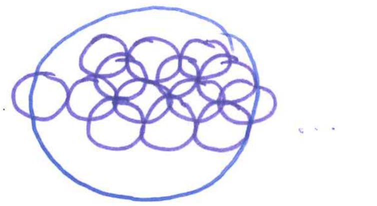
- large positive values are very unlikely $\longrightarrow$ bound prob.
- large negative values: include into definition of "red square".

Lemma 2

Let $\delta > 0$. $P(\max_{|z|\leq R} u(z) \geq R^\delta) \leq CR^2 \exp(-ce^{-R^\delta})$

Proof:

u is shift-invariant & $\underbrace{B(0,R)}_{\text{disk of radius } R}$ can be covered by CR^2 copies of the unit disk:



$$\Longrightarrow P(\max_{|z|\leq R} u(z) > R^\delta)$$
$$\leq CR^2 P(\max_{|z|\leq 1} u(z) > R^\delta)$$
$$\leq CR^2 P(\max_{|z|\leq 1} |f(z)| > \exp(R^\delta))$$
$$\leq CR^2 P\left(\sum_u \frac{1}{\sqrt{u!}}|\xi_u| > \exp(R^\delta)\right)$$
$$\underset{\text{Lemma A.1}}{\leq} CR^2 \cdot 2\exp\left(-\frac{1}{2}\left(\sum_u \frac{1}{\sqrt{u!}}\right)^{-2}\exp(2R^\delta)\right)$$

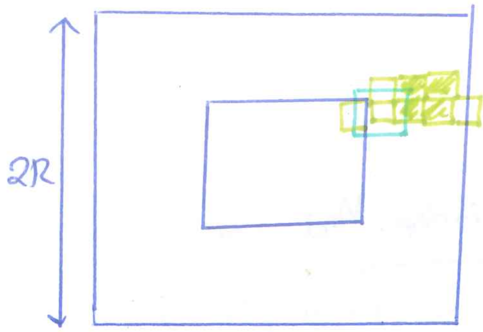
$\square$

4. Decoupling of distant blocks, Proof of Theorem

Let $r = \sqrt{\log(R)}$.

Consider the partition of $\mathbb{C}$ into blocks $Q(w_j, r)$ of side-length r, set

$$J = \{j : Q(w_j, 2r) \subseteq Q(0,2R) \setminus Q(0,R)\}$$



$\# J \leq 4 \left(\frac{R}{r}\right)^2.$

<u>Def.</u>: $j \in J$. $Q(w_j, r)$ is called <u>bad</u> if $\exists$ curve γ joining $\partial Q(w_j, r) \longleftrightarrow \partial Q(w_j, 2r)$ s.t.

$$\text{(i)} \quad u < -R^\delta \text{ everywhere on } \Gamma$$

$$\text{or (ii)} \quad \int_\gamma |\nabla u| |dz| \underbrace{< r R^{2\delta - 1}}_{\text{"noticable presence of red squares"}}$$

otherwise $Q(w_j, r)$ is <u>good</u>.

Intuition: A long gradient curve leaves a trace of bad blocks!

Let $A = A(M = 1)$ from Prop. 1 and $J' \subseteq J$ s.t. $|w_i - w_j| \geq 2Ar$ $\forall i \neq j$ in J'.

$\Longrightarrow T_{w_i} f = f_i + h_i$, f_i indep. G.E.F., h_i very small on $|z| \leq 2r$.

Proposition 3

$\exists$ events $\{\Omega_j\}_{j \in J'}$ dep. only on f_j (thus indep.): and an event Ω_* s.t.:

- $P(\Omega_j) \leq \frac{1}{\sqrt{R}}$
- $P(\Omega_*) \leq C \cdot e^{-cR^4}$

s.t. $\{Q(w_j, r) \text{ bad}\} \subseteq \Omega_j \cup \underbrace{\Omega_*}_{\hat{=} \text{ large values of } h_j}$

<u>Proof (idea)</u>:

$$\Omega_* = \{\max_{j \in J'} \max_{|z| \leq 2r} |h_j(z)| > R^{-4}\}, \quad R^{-4} = e^{-(2r)^2}$$

$$\left(\xRightarrow{\text{Prop. 1}} P(\Omega_*) \leq C \cdot \# J' \cdot \exp(-c \exp((2r)^2)) \leq Ce^{-cR^4} \right)$$

Now partition $\mathbb{C}$ into squares of side-length R^{-1}. For $j \in J'$:
w_{ju} = centers of squares of side-length $\frac{1}{R}$ in $Q(w_j, 2r)$

We call $Q(w_{ju}, \frac{1}{R})$ red if $\exists z \in Q(w_{ju}, \frac{1}{R})$:

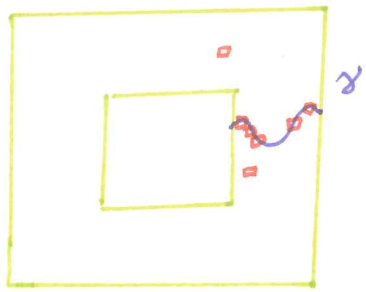
(i) $u(z) < -R^{\delta}$

or (ii) $|\nabla u(z)| < 3 \cdot R^{2\delta - 1}$

Lemma 4

If $\#\{u : Q(w_{ju}, \frac{1}{R}) \text{ is red}\} \leq R$, then $Q(w_j, r)$ is good.

Proof:



Let γ be any curve connecting $\partial Q(w_j, r)$, $\partial Q(w_j, 2r)$.

The red squares cannot cover γ entirely (sum of sides $= 1 < \frac{r}{2}$)

$\Longrightarrow$ $\exists$ points on γ: $u > -R^{\delta}$ ((i) of def. of bad fails)

Let γ_* = part of $\gamma \cap (Q(w_j, 2r - \frac{1}{R}) \setminus Q(w_j, r + \frac{1}{R}))$ not covered by red squares:

$$\int_\gamma |\nabla u||dz| \geq \int_{\gamma_*} |\nabla u||dz| \underset{\substack{\uparrow \\ \text{not} \\ \text{red}}}{\geq} 3R^{2\delta-1} \cdot \text{Length}(\gamma_*)$$

$$\geq 3R^{2\delta-1}\Big(\frac{r}{2} - \underbrace{1}_{\substack{\text{max. lngth of} \\ \text{red squares}}} - \underbrace{\frac{2}{R}}_{\text{boundaries}}\Big) \geq R^{2\delta-1} r$$

for $R \gg 1$. Thus, $Q(w_j, r)$ cannot be bad. $\square$

The rest of the proof is to show that

$\#\{u : Q(w_{ju}, \frac{1}{R}) \text{ is red}\} > R$ is unlikely!

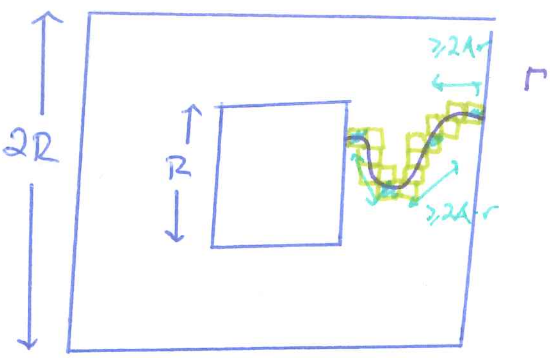
($\nabla u(w)$ can be repr. in terms of $(T_n f)'$; $(T_n f)$ ...)

This corresponds to the calculation ($\square$)

$$P(|\nabla u(w)| < \tfrac{1}{R}) = \ldots \leq CR^{-2}$$

Proof of main Theorem

Let Γ be a gradient curve connecting $\partial Q(0,R) \longleftrightarrow \partial Q(0,2R)$.
By Lemma 2: Assume wlog
$u \leq R^\delta$ on Γ.
Consider $Q(w_j, r)$, $j \in J$ intersecting Γ,
and L = number of these blocks.



$$\frac{\left(\frac{R}{2} - 2r\right)}{r} \leq L \leq 4\left(\frac{R}{r}\right)^2$$

Lemma 5

$\leq \frac{8R^{1-\delta}}{r}$ of these are good.

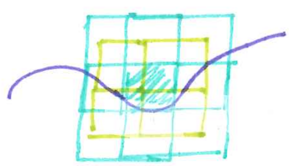
Proof:

Consider a good block $Q(w_j, r)$ and let γ_j be a connected part of $\Gamma \cap (Q(w_j, 2r) \setminus Q(w_j, r))$ joining $\partial Q(w_j, r) \longleftrightarrow \partial Q(w_j, 2r)$.
If $Q(w_j, 2r)$ has no point of Γ with $u = -R^\delta$ (can be only one), then $u \geq -R^\delta$ on γ_j
and number of ~~blocks~~ good blocks containing it ≤ 4

$$\Longrightarrow \int_{\gamma_j} |\nabla u(z)| \cdot |dz| \geq r \cdot R^{2\delta - 1}$$

$\hat{N}$ = # good blocks.

Each point of Γ belongs to ≤ 4 ~~squares~~ blocks of size $2r$



$$\Longrightarrow \hat{N} r R^{2\delta-1} \leq \sum_{j:\, Q(w_j, r) \text{ good}}' \int_{\gamma_j} |\nabla u(z)| |dz|$$

$$\leq 4 \int_{\Gamma \cap \{|u| < R^\delta\}} |\nabla u(z)| \cdot |dz| \leq 8R^\delta. \qquad \square$$

Since $\frac{8R^{1-\delta}}{r}$ grows slower than $\frac{\left(\frac{R}{2} - 2r\right)}{r}$, $\exists$ connected family of blocks connecting $\partial Q(0,R)$, $\partial Q(0,2R)$ of cardinality $L \geq C\frac{R}{r}$, $\mathcal{F}' \subseteq \mathcal{F}$ subfamily of which are bad, with $\# \mathcal{F}' = \frac{L}{2}$.

Now choose within $\mathcal{F}'$ $c\frac{L}{A^2}$ blocks s.t. distances of the centers are $\geq 2Ar$.

$$\overset{\text{Prop 3}}{\Longrightarrow} P(\exists \mathcal{F}' \subseteq \mathcal{F} \text{ as above})$$

$$\leq \underbrace{\left(\frac{1}{\sqrt{R}}\right)^{cA^{-2}L}}_{\text{indep. of } \Omega_j,\, j\in J'} + Ce^{-cR^4} \leq e^{-cA^{-2}L\log R} + Ce^{-cR^4}$$

$$\leq 2e^{-cL\log R},$$

since $L \leq 4\left(\frac{R}{r}\right)^{\alpha}$. □

The proof is concluded once we observe that there are $\leq C\left(\frac{R}{r}\right)^2 e^{CL}$ connected families of boxes of size L and $\leq 2^L$ subfamilies of such families. □