Gravitational allocation to uniform points on the sphere

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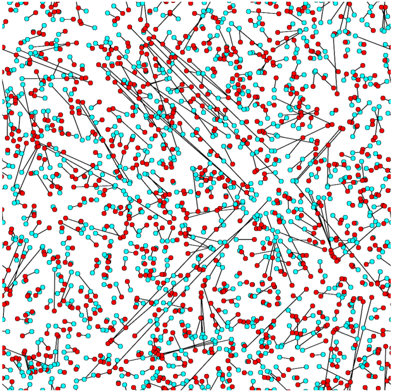
Based on joint work with

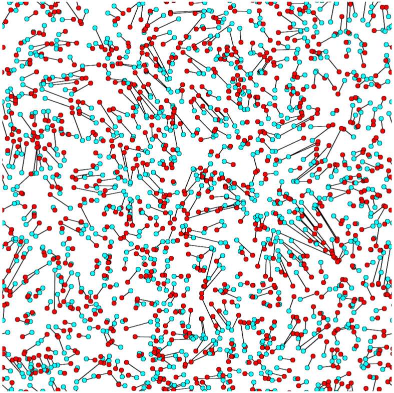
Yuval Peres
Microsoft Research

Alex Zhai
Stanford University

Matchings



Stable matching



Optimal matching

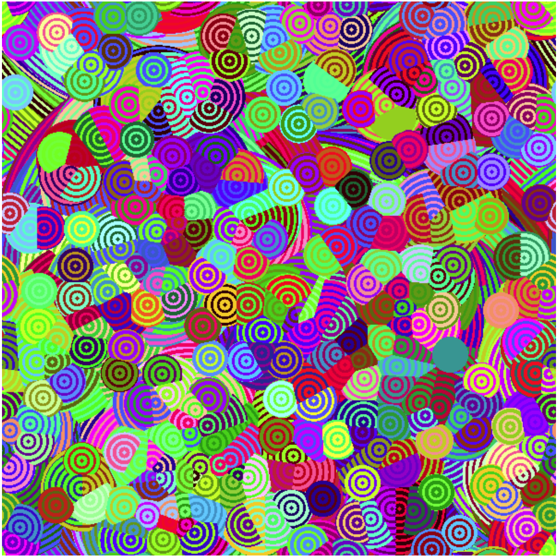
Matching of n blue points and n red points sampled uniformly and independently from the torus.

Fair allocations

- For $n \in \mathbb{N}$ let $\mathbb{S}^2_n \subset \mathbb{R}^3$ be the sphere centered at the origin with radius chosen such that the total surface area $\lambda(\mathbb{S}^2_n)$ equals n.
- For any set $\mathcal{L} \subset \mathbb{S}^2_n$ consisting of n points called *stars*, we say that a measurable function $\psi : \mathbb{S}^2_n \to \mathcal{L} \cup \{\infty\}$ is a *fair allocation* if it satisfies the following:

$$\lambda(\psi^{-1}(\infty)) = 0, \qquad \lambda(\psi^{-1}(z)) = 1, \quad \forall z \in \mathcal{L}. \tag{1}$$

- In other words, a fair allocation is a way to divide $\mathbb{S}^2_n$ into n cells of measure 1 (up to a set of measure 0), with each cell associated to a distinct star in $\mathcal{L}$.

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Notices

of the American Mathematical Society

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AMS
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Dividing the sphere into n equal areas about n given points (see page 446).

Allocation rule

- Let $\mathcal{L} \subset \mathbb{S}^2_n$ be a random collection of n points.
- An **allocation rule** is a measurable mapping $\mathcal{L} \to \psi_\mathcal{L}$ such that
  - $\psi_\mathcal{L}$ is a fair allocation of λ to $\mathcal{L}$ a.s., and
  - $\mathcal{L} \mapsto \psi_\mathcal{L}$ is rotation-equivariant, i.e., $\mathbb{P}$-a.s., for any $x \in \mathbb{S}^2_n$ and any rotation map ϕ, we have $\psi_{\phi(\mathcal{L})}(\phi(x)) = \phi(\psi_\mathcal{L}(x))$.
- We are interested in minimizing $|\psi_\mathcal{L}(x) - x|$ for $x \in \mathbb{S}^2_n$.

Gravitational allocation

Potential and field: $U(x) = \sum_{z \in \mathcal{L}} \log|x - z|, \qquad F(x) = -\nabla_S U(x),$

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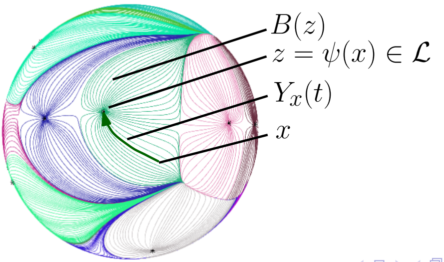
Flow lines: $\frac{dY_x}{dt}(t) = F(Y_x(t)), \quad Y_x(0) = x, \quad t \in [0, \tau_x],$



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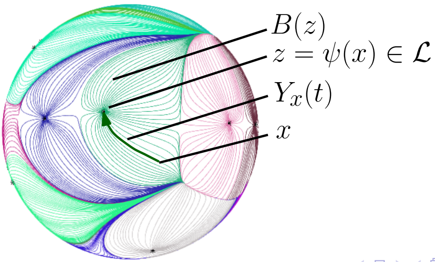
Basin of attraction: $B(z) = \{x \in \mathbb{S}^2_n : \lim_{t \uparrow \tau_x} Y_x(t) = z\}, \ z \in \mathcal{L},$



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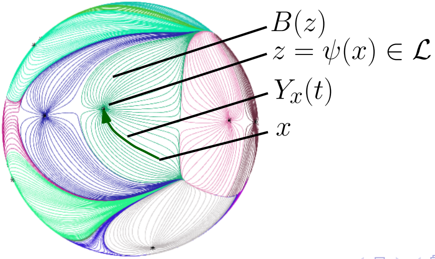
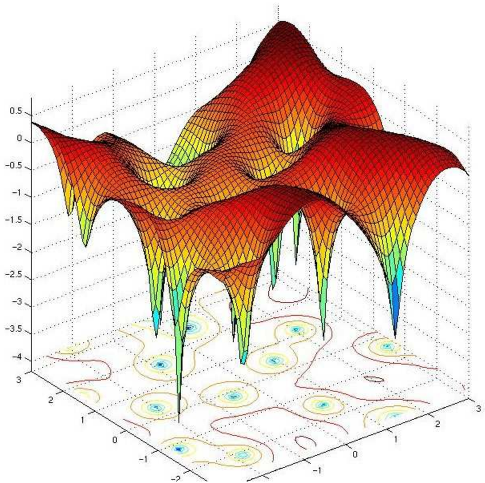
Basin of attraction: $B(z) = \{x \in \mathbb{S}^2_n : \lim_{t \uparrow \tau_x} Y_x(t) = z\}, \ \ z \in \mathcal{L},$

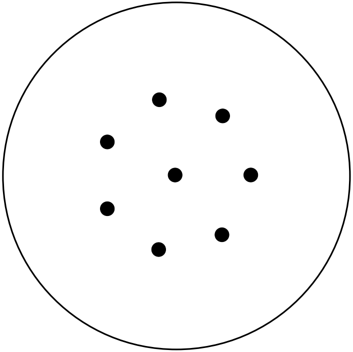
Terminal point: $\psi(x) = \begin{cases} z & \text{if } x \in B(z) \text{ for } z \in \mathcal{L}, \\ \infty & \text{if } x \notin \bigcup_{z \in \mathcal{L}} B(z). \end{cases}$



Gravitational potential

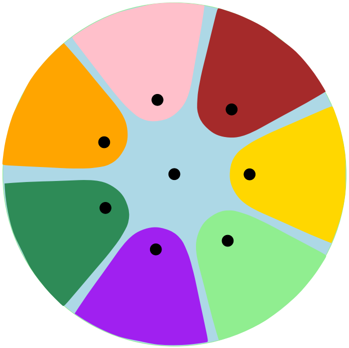


Examples



One point on the north pole, surrounded by seven other points

Examples



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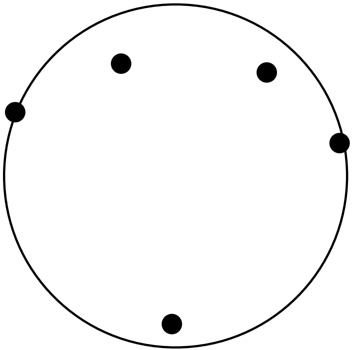

One point on the south pole and seven points in the northern hemisphere

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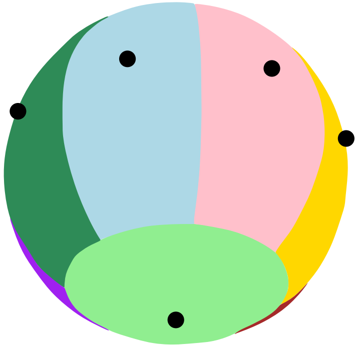


One point on the south pole and seven points in the northern hemisphere

Why is the allocation fair?

Divergence theorem, assuming $B(z_0)$ has a piecewise smooth boundary for $z_0 \in \mathcal{L}$:

$$\int_{B(z_0)} \Delta_S U \, d\lambda = -\int_{\partial B(z_0)} F(x) \cdot \mathbf{n} \, ds. \tag{2}$$

Observe that if $\lambda(\mathbb{S}^2) = A$, then

$$\Delta_S \log|x - z| = 2\pi\delta_z - \frac{2\pi}{A} \qquad \Rightarrow \qquad \Delta_S U = 2\pi \sum_{z \in \mathcal{L}} \delta_z - \frac{2\pi n}{A}.$$

Since $F(x) \cdot \mathbf{n} = 0$ for $x \in \partial B(z_0)$, we get by insertion into (2) that

$$2\pi - \frac{2\pi n}{A} \lambda(B(z_0)) = 0.$$

Thus $\lambda(B(z_0)) = \frac{A}{n}$ as claimed.

Main result

Let $\mathcal{L}$ be a collection of $n \geq 2$ points chosen uniformly at random from $\mathbb{S}^2_n$.

Theorem (H.-Peres.-Zhai)

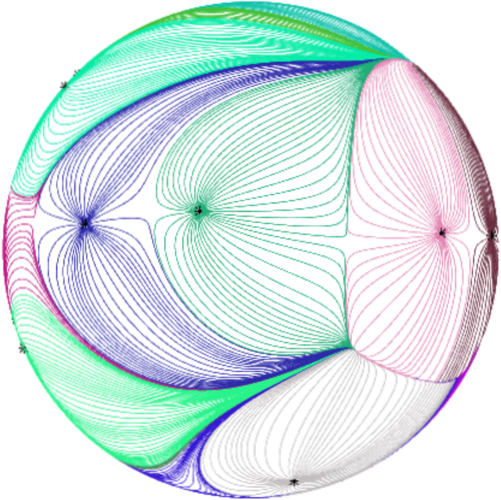
For any $p > 0$ there is a constant $C_p > 0$ such that for any $x \in \mathbb{S}^2_n$

$$\mathbb{P}\left[|\psi(x) - x| > r\sqrt{\log n}\right] \leq C_p r^{-p}.$$

In particular, there is a universal constant $C > 0$ such that for any $x \in \mathbb{S}^2_n$,

$$\mathbb{E}[|\psi(x) - x|] \leq C\sqrt{\log n}.$$

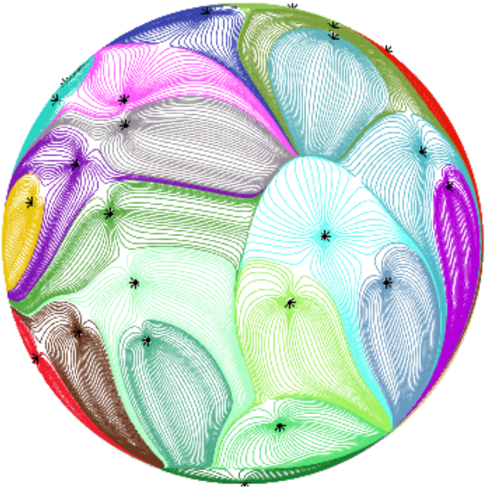
Examples



$n = 15$

Simulation based on code written by Manjunath Krishnapur

Examples

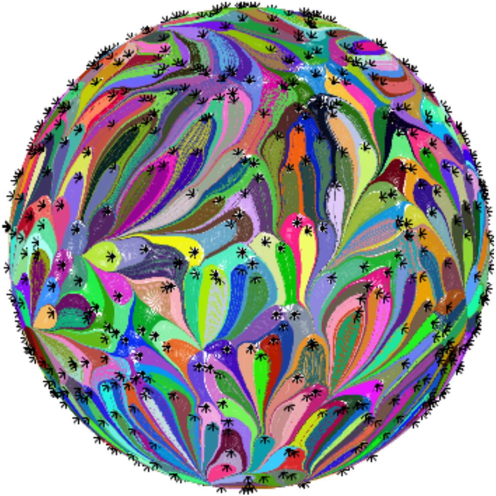


$n = 40$

Examples

$n = 200$

Examples



$n = 750$

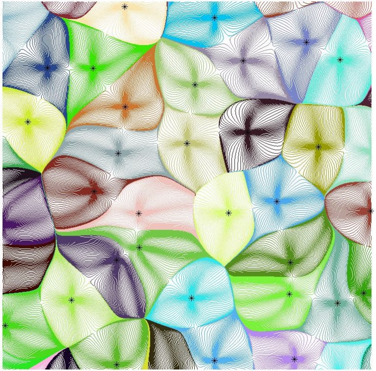
Earlier results on fair allocations

Let $Y = \text{diam}(B(\psi(x)))$ denote the **diameter** of the basin containing x.

In the following settings $\mathbb{P}[Y > R]$ decays superpolynomially in R:

- $\mathcal{L} \subset \mathbb{C}$ the zero set of a Gaussian Entire Function f; potential $U = \log|f| - |z|^2/2$ (Nazarov-Sodin-Volberg'07)
- $\mathcal{L} \subset \mathbb{R}^d$, $d \geq 3$, unit intensity Poisson point process; gravitational field F (Chatterjee-Peled-Peres-Romik'10)

Remark: Gravitational allocation to a unit intensity Poisson point process in $\mathbb{R}^2$ is not well-defined.



Gravitational allocation to the zero set of the Gaussian Entire Function

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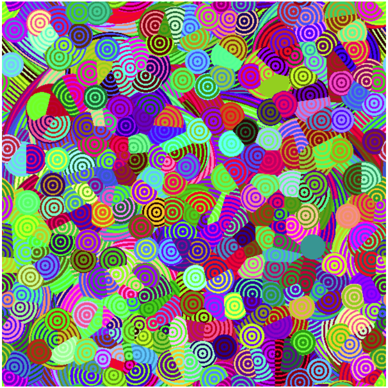

Stable marriage allocation (simulation by A. Holroyd)

Application: matchings

- Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be sampled uniformly and independently at random from $\mathbb{S}_n^2$ for $n \geq 2$.
- A **matching** of $\mathcal{A}$ and $\mathcal{B}$ is a bijective function $\varphi : \mathcal{A} \to \mathcal{B}$.

Corollary (H.-Peres.-Zhai)

We can use gravitational allocation to define a matching, such that for a universal constant $C > 0$,

$$\mathbb{E}[X] \leq C\sqrt{\log n}, \qquad X := \frac{1}{n}\sum_{i=1}^{n} |\varphi(a_i) - a_i|.$$

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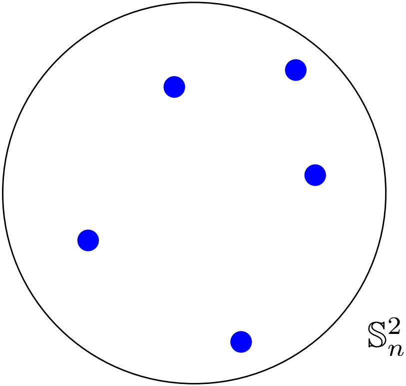
Ajtai-Komlós-Tusnády'84: There exists a universal constant $C > 1$ such that for the optimal matching φ of $2n$ points in $[0, \sqrt{n}]^2$,

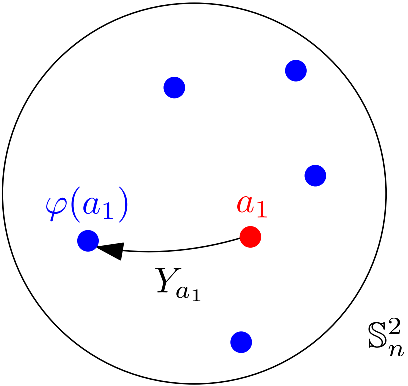
$$\lim_{n\to\infty} \mathbb{P}\left[C^{-1}\sqrt{\log n} \leq X \leq C\sqrt{\log n}\right] = 1.$$

Proof Matching Corollary



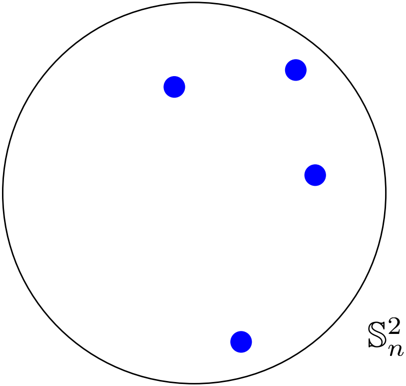
The points of $\mathcal{B}$ define a gravitational potential.

Proof Matching Corollary



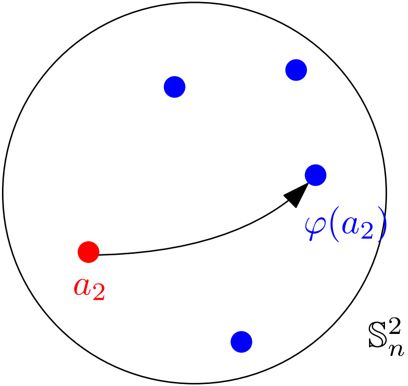
Pick the points of $\mathcal{A}$ one by one.
By the main theorem, $\mathbb{E}[|\varphi(a_1) - a_1|] \le C\sqrt{\log n}$.

Proof Matching Corollary



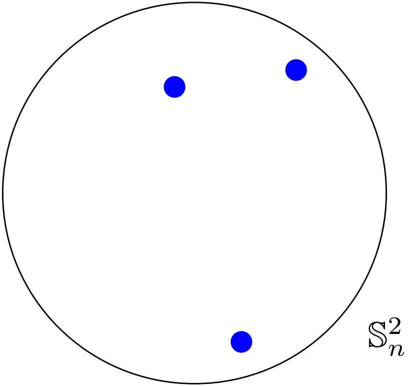
Note that the remaining points are **uniformly distributed**.

Proof Matching Corollary



By the main theorem, $\mathbb{E}[|\varphi(a_2) - a_2|] \leq C\sqrt{\frac{n}{n-1}}\sqrt{\log(n-1)}$.

Proof Matching Corollary



The remaining points are again uniformly distributed.
Repeat the procedure until all points are matched.

Proof Matching Corollary

- Combining the above bounds, we get

$$X = \frac{1}{n}\sum_{k=1}^{n} |\varphi(a_k) - a_k|,$$

$$\mathbb{E}[X] \le \frac{C}{n}\sum_{k=1}^{n}\sqrt{\frac{n}{k}}\sqrt{1+\log k} \le C_1\sqrt{\log n},$$

as claimed.

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- The matching algorithm is **online**: $\varphi(a_k)$ depends only on $a_1, \ldots, a_k$ and $b_1, \ldots, b_n$.

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Greedy/stable matching

- Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be sampled uniformly and independently at random from $\mathbb{S}^2_n$ for $n \in \mathbb{N}$.
- Define a matching $\varphi : \mathcal{A} \to \mathcal{B}$ by iteratively matching closest pairs.
- Define $Y = |\varphi(a_1) - a_1|$. What is $\mathbb{E}[Y]$?

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- Holroyd-Pemantle-Peres-Schramm'09: For $\mathcal{A}$ and $\mathcal{B}$ Poisson point processes in $\mathbb{R}^2$ and $\widehat{Y} = |\varphi(a) - a|$ for a typical point $a \in \mathcal{A}$,

$$\mathbb{P}[\widehat{Y} > r] \leq C_1 r^{-0.496\dots}.$$

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- Transferring to $\mathbb{S}^2_n$,

$$\mathbb{E}[Y] \leq C_1 \int_0^{\sqrt{n}} r^{-0.496\dots}\, dr = C_2 n^{0.252\dots}.$$

Allocation to Gaussian random polynomial

- For $\zeta_1, \ldots, \zeta_n$ independent standard complex Gaussians,

$$p(z) = \sum_{k=0}^{n} \zeta_k \frac{\sqrt{n(n-1)\cdots(n-k+1)}}{\sqrt{k!}} z^k.$$

- Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the roots of p.
- Let $\mathcal{L} \subset \mathbb{S}_n^2$ correspond to $\lambda_1, \ldots, \lambda_n$ via stereographic projection.
- Let $\psi : \mathbb{S}_n^2 \to \mathcal{L}$ define gravitational allocation to $\mathcal{L}$.

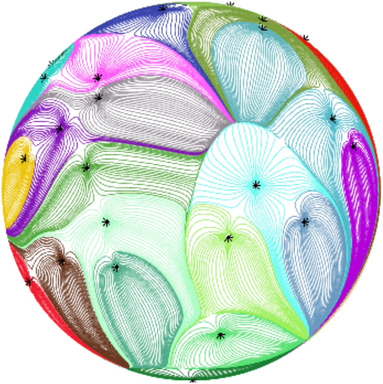
Proposition (H.-Peres-Zhai)

For any fixed $x \in \mathbb{S}_n^2$,

$$\mathbb{E}\left[|x - \psi(x)|\right] = \Theta(1).$$

Allocation to Gaussian random polynomial

Gaussian random polynomial



Uniform points

Allocation to Gaussian random polynomial (cont.)

The average distance travelled can be expressed in terms of the average force:

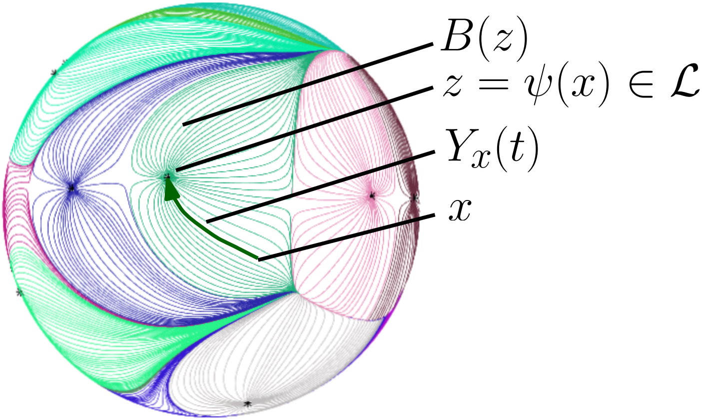
$$\int_{\mathbb{S}^2_n} \int_0^{\tau_x} |F(Y_x(t))|\, dt\, d\lambda_n(x) = \frac{1}{2\pi} \int_{\mathbb{S}^2_n} |F(x)|\, d\lambda_n(x) \qquad \Rightarrow$$

$$\mathbb{E}[|x - \psi(x)|] \leq \frac{1}{2\pi}\mathbb{E}|F(x)|.$$

We express the force in terms of the coefficients of our Gaussian random polynomial:

$$F(x) = \sqrt{\frac{\pi}{n}} \sum_{k=1}^{n} \overline{\lambda}_k = \sqrt{\frac{\pi}{n}} \cdot \frac{\overline{\zeta}_1 \cdot \sqrt{n}}{\overline{\zeta}_0 \cdot 1} = \sqrt{\pi} \cdot \frac{\overline{\zeta}_1}{\overline{\zeta}_0}.$$

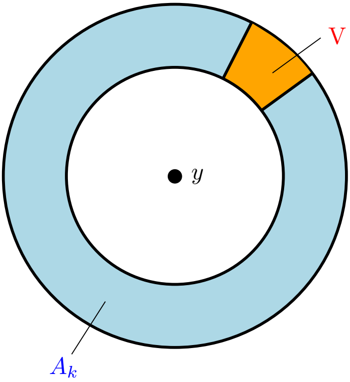
Proof main theorem



Goal: $\mathbb{E}[\|\psi(x) - x\|] \leq C\sqrt{\log n}.$

Force bound

For $V \subset \mathbb{S}_n^2$ let $F(y \mid V)$ denote the force exterted by points $z \in \mathcal{L} \cap V$.



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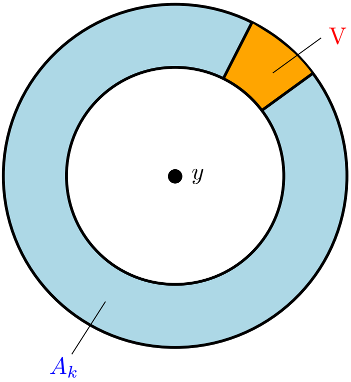
For $k \in \{1, 2, \ldots, \lfloor \frac{1}{10}\sqrt{n} \rfloor\}$ and $\lambda(V) = 1$,

$$\mathrm{Var}[F(y \mid V)] = \Theta(k^{-2}),$$
$$\mathrm{Var}[F(y \mid A_k)] = \Theta(k^{-1}),$$
$$\mathrm{Var}[F(y \mid \cup_{k=1}^{\sqrt{n}/10} A_k)] = \Theta(\log n).$$



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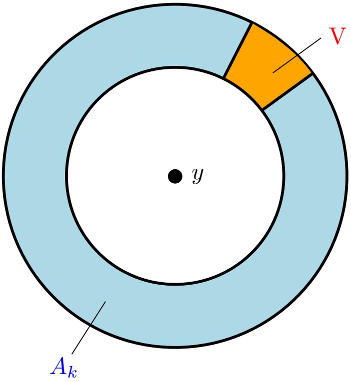

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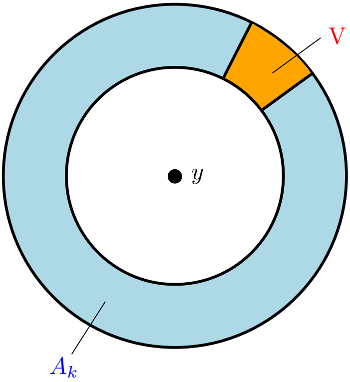
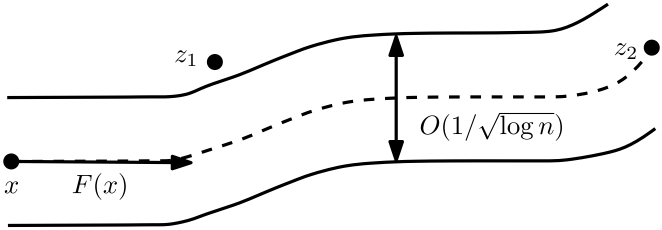


Heuristic proof main theorem

- Recall: $|F(y;r_n)| = O(1/r_n)$ with high probability uniformly in y.
  - $r_n = 1/\sqrt{\log n}$.
  - $F(y;r_n)$ is the force at y exerted by points $z \in \mathcal{L}$ satisfying $|y-z| > r_n$.
- The point x travels until the force from nearby particles dominates $F(Y_x(t);r_n)$, i.e., at most until $|Y_x(t) - z| = cr_n$ for some $z \in \mathcal{L}$ and $c \ll 1$ constant.
- Therefore $|\psi(x) - x| = O(1/(cr_n)) = O(\sqrt{\log n})$ with high probability.



Proof idea main theorem

Note that since

$$\psi(x) = Y_x(\tau_x) = \int_0^{\tau_x} F(Y_x(t))\,dt + x,$$

it is sufficient to bound the following to bound $|\psi(x) - x|$ from above

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We show:

(a) $\mathbb{P}[\tau_x > t] = e^{-2\pi t}$ (see next slide).

(b) By the force bound, if $|F(y)| \gg \sqrt{\log n}$, then y will be swallowed by a point at distance $O(1/\sqrt{\log n})$ with high probability.

Liouville's theorem gives the probability distribution of τ_x

Liouville's Theorem: For M an oriented 2-dimensional Riemannian manifold with volume form $d\alpha$, a smooth vector field F on M, Φ_t the flow induced by F, and Ω an open set with compact closure,

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}_\alpha(\Phi_t(\Omega)) = \int_\Omega \mathrm{div}(F)\, d\alpha.$$

By the following lemma, $\mathbb{P}[\tau_x > t] = e^{-2\pi t}$.

Lemma

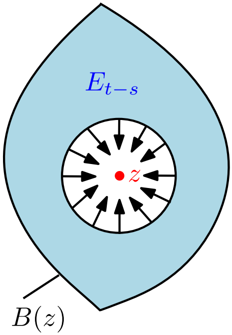
For $z \in \mathcal{L}$ and $t \geq 0$, define

$$E_t = \{x \in B(z) : \tau_x > t\}, \qquad V_t = \lambda(E_t).$$

Then $V_t = e^{-2\pi t} V_0$.

The lemma is proved by applying Liouville's theorem with $F = -\nabla_S U$ and $\Omega = E_{t-s}$:

$$\frac{d}{ds} V_{t-s} = -\int_{E_{t-s}} \Delta_S U\, d\lambda = \int_{E_{t-s}} 2\pi\, d\lambda = 2\pi V_{t-s}.$$



Conjectures for optimal squared matching distance

- Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a collection of n points chosen uniformly and independently at random from the d-dimensional torus $\mathbb{T}^d$ of area n. Let $\varphi : \mathcal{A} \to \mathcal{B}$ denote the matching which minimizes the cost $\frac{1}{n}\sum_{a\in\mathcal{A}} |\varphi(a) - a|^2$.

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$$\mathbb{E}\left[\frac{1}{n}\sum_{a\in\mathcal{A}} |\varphi(a) - a|^2\right] \sim \begin{cases} \frac{1}{6}n + c_1 & \text{if } d = 1, \\ \frac{1}{2\pi}\log n + c_2 & \text{if } d = 2, \\ c_3 + \frac{\zeta_d(1)}{2\pi^2} n^{-\frac{d-2}{2}} & \text{if } d \geq 3. \end{cases} \quad (3)$$

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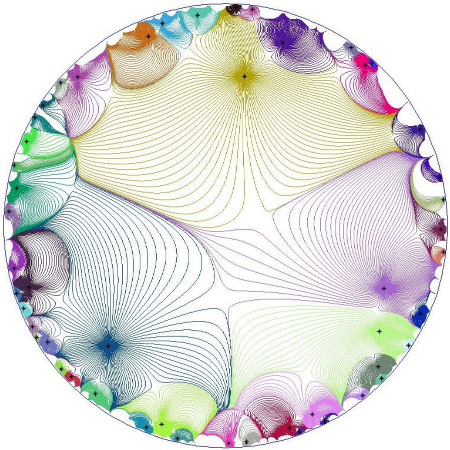
- Ambrosio-Stra-Trevisan'16 established the leading constant $\frac{1}{2\pi}$ for $d = 2$ rigorously. Their analysis suggests that gravitaitonal allocation is asymptotically optimal for the cost function $|\varphi(a) - a|^2$.

Allocation of hyperbolic plane to zeros of Gaussian hyperbolic functions



Intensity of zeros $= 1$ (simulation by J. Ding and R. Peled)

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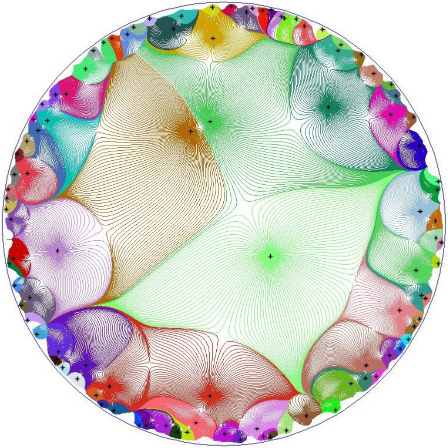

Intensity of zeros = 3 (simulation by J. Ding and R. Peled)

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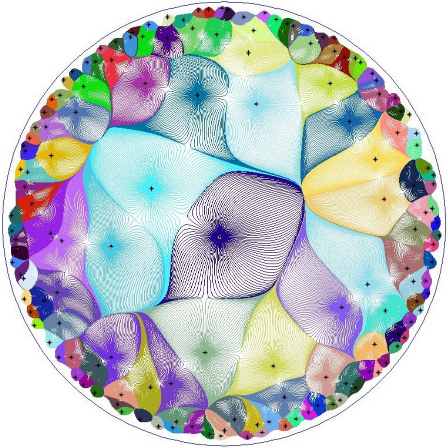
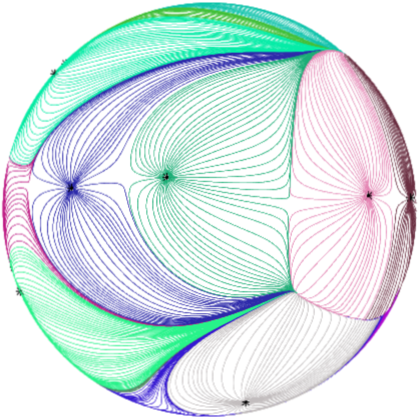


Intensity of zeros = 10 (simulation by J. Ding and R. Peled)

Open problem: electrostatic matching

- Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a collection of n particles on $\mathbb{S}_n^2$ with negative (resp. positive) charge, sampled independently and uniformly at random.
- Assume particles of different (resp. similar) charge attract (resp. repulse) each other.
- Does this define a matching of $\mathcal{A}$ and $\mathcal{B}$ a.s.? What is the expected average distance between matched particles?



Thanks!