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Numerical Analysis of High-Dimensional Problems

ETH Zürich
D-MATH

Exercise 1

Problem 1.1 inf-sup condition in L^p

Let $q \in (1, \infty)$ and $1/p + 1/q = 1$. In the following we consider the one dimensional Sobolev spaces $W_0^{1,r} = W_0^{1,r}([0,1])$, $r \in \{p, q\}$, on the interval $[0,1]$.

(1.1a) Prove that

$$\inf_{0 \neq u \in W_0^{1,p}} \sup_{0 \neq v \in W_0^{1,q}} \frac{\int_0^1 u'v' \,\mathrm{d}x}{\|u'\|_{L^p} \|v'\|_{L^q}} = 1. \tag{1.1.1}$$

Theorem (Fredholm alternative). *Let T be a compact linear operator on a normed space $\mathcal{X}$ into itself. Then either (i) the homogeneous equation*

$$u - Tu = 0$$

has a nontrivial solution $u \in \mathcal{X}$ or (ii) for each $v \in \mathcal{X}$ the equation

$$u - Tu = v$$

has a uniquely determined solution $u \in \mathcal{X}$. Furthermore, in case (ii), the operator $(\mathrm{Id} - T)^{-1}$ whose existence is asserted there is also bounded.

For a proof see Theorem 5.3 in "Elliptic Partial Differential Equations of Second Order", Gilbarg and Trudinger, Springer.

(1.1b) Use subproblem (1.1a) to show that

$$\inf_{0 \neq u \in W_0^{1,p}} \sup_{0 \neq v \in W_0^{1,q}} \frac{\int_0^1 u'v' + uv \,\mathrm{d}x}{\|u'\|_{L^p} \|v'\|_{L^q}} > 0. \tag{1.1.2}$$

HINT: Use Thm. 1.36, the given Theorem ("Fredholm alternative") and the fact that $W_0^{1,q}$ is reflexive.

Problem 1.2 Semilinear Elliptic PDE

Consider the semilinear elliptic PDE

$$-q'' + q^3 = f \quad \text{in } D = (0,1) \text{ with } q(0) = q(1) = 0. \tag{1.2.1}$$

Definition. *We call $\mathcal{R}$ Fréchet differentiable at $q \in \mathcal{X}$ if there exists a bounded, linear operator $A : \mathcal{X} \to \mathcal{Y}'$ such that*

$$\lim_{\|w\|_{\mathcal{X}} \to 0} \frac{\|\mathcal{R}(q+w) - \mathcal{R}(q) - Aw\|_{\mathcal{Y}'}}{\|w\|_{\mathcal{X}}} = 0,$$

and A is the Fréchet derivative of $\mathcal{R}$ at q. We also write $A = D\mathcal{R}(q)$.

(1.2a) Let $\mathcal{R}(q) = -q'' + q^3 - f$, $\mathcal{X} = H_0^1(D)$ and $\mathcal{Y} = H_0^1(D)$. Prove that $\mathcal{R} : \mathcal{X} \to \mathcal{Y}'$ is Fréchet differentiable at $q \in \mathcal{X}$ and write down an expression for the Fréchet derivative $D\mathcal{R}(q)$.

Definition. *A functional $I : \mathcal{X} \to \mathbb{R}$ on a Banach space $\mathcal{X}$ is called coercive if for every sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{X}$,*

$$\|u_k\|_{\mathcal{X}} \to \infty \text{ as } k \to +\infty \text{ implies that } I(u_k) \to +\infty \text{ as } k \to +\infty.$$

For the next Proposition see Thms. 1.5.6, 1.5.8 in 'Semilinear Elliptic PDEs for Beginners' by M. Badiale and E. Serra, Springer – London, 2011.

Proposition. *Let $\mathcal{X}$ be a reflexive Banach space and let $I : \mathcal{X} \to \mathbb{R}$ be continuous, strictly convex and coercive. Then I has a unique minimum in $\mathcal{X}$.*

(1.2b) Prove that for all $f \in \mathcal{Y}'$, there exists a unique solution $q \in \mathcal{X}$ such that $\mathcal{R}(q) = 0$ in $\mathcal{Y}'$ by using critical points of a functional $I : \mathcal{X} \to \mathbb{R}$ such that $DI = \mathcal{R}$.

Hint: Apply the above proposition.

(1.2c) Prove that for $f \in \mathcal{Y}'$ such that $\|f\|_{\mathcal{Y}'}$ is sufficiently small, there exists a locally unique solution $q \in \mathcal{X}$ such that $\mathcal{R}(q) = 0$ in $\mathcal{Y}'$ by verifying the assumptions of Theorem 1.41 in the lecture notes.

HINT: Consider $w \in H_0^1(D)$, which uniquely solves $-w'' = f$. Prove that the solution q lies in a neighborhood of w.

Problem 1.3 Sobolev embedding in one dimension

For $D = (0, 1)$, show that the embedding $H^1(D) \subset C^{0,\frac{1}{2}}(\overline{D})$ is continuous.

Hint: You may assume that the continuous embedding $H^1(D) \subset C^0(\overline{D})$ is already established.

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