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Numerical Analysis of High-Dimensional Problems

ETH Zürich
D-MATH

Exercise 3

Problem 3.1 Bochner spaces as tensor products

Definition *Let X, Y be two real Banach spaces and p, $q \in (1,\infty)$ s.t. $1/p + 1/q = 1$. For $y_1, \ldots, y_n$ set*

$$\mu_q(y_1, \ldots, y_n) := \sup_{\|\psi\|_{Y'} \leq 1} \left(\sum_{i=1}^n |\psi(y_i)|^q \right)^{1/q}, \tag{3.1.1}$$

and for $z \in X \otimes_\alpha Y$ we introduce the norm

$$\|z\|_p := \inf \left\{ \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \mu_q(y_1, \ldots, y_n) \,\middle|\, z = \sum_{i=1}^n x_i \otimes y_i \right\}. \tag{3.1.2}$$

The space $X \otimes_p Y$ is the closure of the algebraic tensor product $X \otimes_\alpha Y$ under this norm.

Let (Ω, Σ, μ) be a finite measure space. We consider the following three natural embeddings

- $\Lambda : L^p(\Omega, \mathbb{R}) \otimes_p Y \to L^p(\Omega, Y)$ via $\Lambda(\sum_{i=1}^n f_i \otimes y_i)(\omega) = \sum_{i=1}^n f_i(\omega) y_i$,
- $\Gamma : L^p(\Omega, Y) \to Y \otimes_p L^p(\Omega, \mathbb{R})$ via $\Gamma(\sum_{i=1}^n \mathbb{1}_{A_i} y_i) = \sum_{i=1}^n y_i \otimes \mathbb{1}_{A_i}$,
- $\Phi : Y \otimes_p L^p(\Omega, \mathbb{R}) \to L^p(\Omega, \mathbb{R}) \otimes_p Y$ via $\Phi(\sum_{i=1}^n y_i \otimes f_i) = \sum_{i=1}^n f_i \otimes y_i$,

where the measure on the L^p spaces is always μ, and $\omega \in \Omega$, $A_i \in \Sigma$, $y_i \in Y$, $f_i \in L^p(\Omega, \mathbb{R})$ for all i. Note that the above embeddings, which are given on dense subsets, are obtained as the unique continuous extensions of the respective maps.

(3.1a) Prove that $\|\Lambda\|_{L(L^p(\Omega,\mathbb{R}) \otimes_p Y, L^p(\Omega, Y))} = 1$.

(3.1b) Prove that $\|\Gamma\|_{L(L^p(\Omega,Y), Y \otimes_p L^p(\Omega,\mathbb{R}))} = 1$.

HINT: Proceed as follows:

- Consider (the dense set of) elements $\sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} y_i$ where $A_i \cap A_j =$ for all $i \neq j$ and the $\alpha_i \in \mathbb{R}$ are such that $\|\alpha_i \mathbb{1}_{A_i}\|_{L^p} = 1$ for all $i = 1, \ldots, n$.
- Show that for such expansions

$$\mu_q(\alpha_1 \mathbb{1}_{A_1}, \ldots, \alpha_n \mathbb{1}_{A_n}) = \sup_{\sum_{i=1}^n \lambda_i^p \leq 1} \left\| \sum_{i=1}^n \lambda_i \alpha_i \mathbb{1}_{A_i} \right\|_{L^p} = 1. \tag{3.1.3}$$

- Conclude that $\|\sum_{i=1}^n y_i \otimes \alpha_i \mathbb{1}_{A_i}\|_{Y \otimes_p L^p(\Omega,\mathbb{R})} \leq \|\sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} y_i\|_{L^p(\Omega,Y)}$.

(3.1c) Assume that Φ is an isometric isomorphism. Prove that

$$L^p(\Omega, Y) = L^p(\Omega, \mathbb{R}) \otimes_p Y \tag{3.1.4}$$

in the sense that there exists an isometric isomorphism.

HINT: Show that $\Lambda\Phi\Gamma = \mathrm{Id}$ and consider $\Phi\Gamma$.

(3.1d) Conclude that $L^p([0,1]) \otimes_p L^p([0,1]) = L^p([0,1]^2)$ (w.r.t. the Lebesgue measure).

Problem 3.2 Quasi optimality of nonlinear FEM

Give a proof of item 2 in the proof of Thm. 1.47 in the lecture notes: Let q^h be a Petrov–Galerkin solution. Recall that $\mathcal{R}_h : \mathcal{X} \to \mathcal{Y}'$, cp. (1.103) from the lecture notes. Show that $\mathcal{R}_h(q^h) = 0$ and that $q_h \in \mathcal{X}^h$ is the only solution $w \in \mathcal{X}$ of $\mathcal{R}_h(w) = 0$.

Problem 3.3 Non-separability of Hölder spaces

Let $\gamma \in (0,1]$ and $k \in \mathbb{N}_0$. Show that the Hölder space $C^{k,\gamma}([0,1])$ is not separable.

HINT: Find an uncountable set of Hölder functions such that their distance is uniformly bounded from below.

Problem 3.4 Hilbert–Schmidt Operators

(3.4a) Let H_1, H_2 be separable Hilbert spaces. Let $(e_k)_{k\geq 1}$ be an ONB of H_1. Recall the Hilbert–Schmidt norm, which is induced by the inner product

$$(S,T)_{\mathrm{HS}} := \sum_{k\geq 1} (Se_k, Te_k)_{H_2}, \quad \forall S,T \in \mathcal{L}_{\mathrm{HS}}(H_1, H_2).$$

Show that the induced norm does not depend on the choice of ONB.

(3.4b) Let H_1, H_2 be separable Hilbert spaces. If $S \in \mathcal{L}(H_2)$ and $T \in \mathcal{L}_{\mathrm{HS}}(H_1, H_2)$, then $ST \in \mathcal{L}_{\mathrm{HS}}(H_1, H_2)$ and

$$\|ST\|_{\mathcal{L}_{\mathrm{HS}}(H_1,H_2)} \leq \|S\|_{\mathcal{L}(H_2)} \|T\|_{\mathcal{L}_{\mathrm{HS}}(H_1,H_2)}$$

Problem 3.5 Nuclear Operators

Let H_1, H_2, H_3 be separable Hilbert spaces.

(3.5a) If $S \in \mathcal{L}_{\mathrm{N}}(H_2, H_3)$ and $T \in \mathcal{L}(H_1, H_2)$, then $ST \in \mathcal{L}_{\mathrm{N}}(H_1, H_3)$ and

$$\|ST\|_{\mathcal{L}_{\mathrm{N}}(H_1,H_3)} \leq \|S\|_{\mathcal{L}_{\mathrm{N}}(H_2,H_3)} \|T\|_{\mathcal{L}(H_1,H_2)}$$

(3.5b) If $S \in \mathcal{L}(H_2, H_3)$ and $T \in \mathcal{L}_{\mathrm{N}}(H_1, H_2)$, then $ST \in \mathcal{L}_{\mathrm{N}}(H_1, H_3)$ and

$$\|ST\|_{\mathcal{L}_{\mathrm{N}}(H_1,H_3)} \leq \|S\|_{\mathcal{L}(H_2,H_3)} \|T\|_{\mathcal{L}_{\mathrm{N}}(H_1,H_2)}$$

(3.5c) If $S \in \mathcal{L}(H_1, H_2)$ and $T \in \mathcal{L}(H_2, H_1)$, and if either S or T is of trace class, then $ST \in \mathcal{L}_{\mathrm{N}}(H_1)$ and $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$.

(3.5d) Verify the set inclusions

$$\mathcal{L}_{\mathrm{N}}(H_1, H_2) \subset \mathcal{L}_{\mathrm{HS}}(H_1, H_2) \subset \mathcal{K}(H_1, H_2) \subset \mathcal{L}(H_1, H_2).$$

Problem 3.6 Tensors

(3.6a) Let X, Y be two Banach spaces. Show that for every finite linear combination $0 \neq \sum_{i=1}^n x_i \otimes y_i \in X \otimes_\alpha Y$ there exists $m \leq n$ and two sets of linearly independent vectors $\{a_i \,|\, i = 1, \dots, m\} \subseteq X$, $\{b_i \,|\, i = 1, \dots, m\} \subseteq Y$ such that $\sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^m a_i \otimes b_i$.

(3.6b) Let X, Y, V, W be Hilbert spaces and let $A \in \mathcal{L}(X, V)$, $B \in \mathcal{L}(Y, W)$ be bounded linear opertors. Show that $\|A \otimes B\|_{\mathcal{L}(X \otimes_\eta Y, V \otimes_\eta W)} \leq \|A\|_{\mathcal{L}(X,V)} \|B\|_{\mathcal{L}(Y,W)}$.

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