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Numerical Analysis of High-Dimensional Problems

ETH Zürich
D-MATH

Exercise 4

Problem 4.1 ℓ^p-summability of gpc coefficients

We consider the affine parametric operator equation on a bounded Lipschitz domain $D \subset \mathbb{R}^d$ for all $\boldsymbol{y} \in U = [-1,1]^{\mathbb{N}}$

$$\begin{cases} -\mathrm{div}(u(\boldsymbol{y})\nabla q(\boldsymbol{y})) = f, \\ q(\boldsymbol{y})\Big|_{\partial D} = 0, \end{cases}$$

where $u(\boldsymbol{y}) := \overline{u} + \sum_{j\geq 1} y_j \psi_j$ such that $\overline{u}, \psi_j \in L^\infty(D)$ with $0 < \mathrm{essinf}_{x\in D}\, \overline{u}(x)$ and for some $r > 0$

$$\sum_{j\geq 1} |\psi_j(x)| \leq \overline{u}(x) - r \quad \text{for a.e. } x \in D\ . \tag{4.1.1}$$

The variational formulation reads: for every $\boldsymbol{y} \in U$ find $q(\boldsymbol{y}) \in V := H_0^1(D)$ s.t.

$$\int_D u(\boldsymbol{y})\nabla q(\boldsymbol{y}) \cdot \nabla v \mathrm{d}x = {}_{V'}\langle f, v\rangle_V, \quad \forall v \in V.$$

Part 1:

We recall the Taylor series of the parametric solution

$$q(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} t_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}}, \quad \text{where} \quad t_{\boldsymbol{\nu}} = \frac{1}{\boldsymbol{\nu}!}(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q(\boldsymbol{y}))\Big|_{\boldsymbol{y}=0} \quad \text{for all } \boldsymbol{\nu} \in \mathcal{F}.$$

For $p \in (0,2)$ set $p' = \frac{2p}{2-p}$ and assume that there exists $\boldsymbol{\rho} \in (1,\infty)^{\mathbb{N}}$ such that

$$\sum_{j\geq 1} \rho_j |\psi_j(x)| \leq \overline{u}(x) - r \quad \text{for a.e. } x \in D \tag{4.1.2}$$

and such that $\boldsymbol{\rho}^{-1} \in \ell^{p'}(\mathbb{N})$. The goal of this exercise is to prove under this conditions that $(\|t_{\boldsymbol{\nu}}\|_V)_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$. We will prove this statement in several steps.

(4.1a) Prove that the condition in (4.1.1) is equivalent to:

$$\eta := \left\| \frac{\sum_{j\geq 1} |\psi_i|}{\overline{u}} \right\|_{L^\infty(D)} < 1\ .$$

(4.1b) Prove the following recurrence relation of the Taylor coefficients: for all $0 \neq \boldsymbol{\nu} \in \mathcal{F}$ it holds that

$$\int_D \overline{u} \nabla t_{\boldsymbol{\nu}} \cdot \nabla v \,\mathrm{d}x = - \sum_{j \in \mathrm{supp}(\boldsymbol{\nu})} \int_D \psi_j \nabla t_{\boldsymbol{\nu} - e_j} \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in V \ .$$

Define for all $v \in V$ the energy norm $\|v\|_{\overline{u}}$ by

$$\|v\|_{\overline{u}} := \Big(\int_D \overline{u} |\nabla v|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \ .$$

(4.1c) Set $\beta := \frac{\eta}{2-\eta}$ and prove that for all $k \in \mathbb{N}$ it holds that

$$\sum_{|\boldsymbol{\nu}|=k} \|t_{\boldsymbol{\nu}}\|_{\overline{u}}^2 \leq \beta \sum_{|\boldsymbol{\nu}|=k-1} \|t_{\boldsymbol{\nu}}\|_{\overline{u}}^2 \ .$$

HINT: Apply the claims of (4.1b) and (4.1a).

(4.1d) Prove that

$$\sum_{\boldsymbol{\nu} \in \mathcal{F}} \|t_{\boldsymbol{\nu}}\|_V^2 \leq \frac{(2-\eta)\|\overline{u}\|_{L^\infty(D)}}{(2-2\eta)(\operatorname{ess\,inf}_{x \in D} \overline{u}(x))^3} \|f\|_{H^{-1}(D)}^2 < \infty \ .$$

HINT: Apply the claim of (4.1c).

(4.1e) Prove that the condition in (4.1.2) is equivalent to:

$$\left\| \frac{\sum_{j \geq 1} \rho_j |\psi_i|}{\overline{u}} \right\|_{L^\infty(D)} < 1 \ .$$

(4.1f) Prove under the specified conditions that $(\boldsymbol{\rho}^{\boldsymbol{\nu}} \|t_{\boldsymbol{\nu}}\|_V)_{\boldsymbol{\nu} \in \mathcal{F}} \in \ell^2(\mathcal{F})$.

HINT: Combine the claims of (4.1d) and (4.1e).

(4.1g) Finally prove using the property $\boldsymbol{\rho}^{-1} \in \ell^{p'}(\mathbb{N})$ that $(\|t_{\boldsymbol{\nu}}\|_V)_{\boldsymbol{\nu} \in \mathcal{F}} \in \ell^p(\mathcal{F})$.

HINT: Apply the claim of (4.1f) and use the Hölder inequality.

Part 2:

Consider the tensorized Legendre polynomials $L_{\boldsymbol{\nu}}(\boldsymbol{y}) = \prod_{j \geq 1} L_{\nu_j}(y_j)$, where L_k denotes the Legendre polynomial on $[-1,1]$, which is normalized in $L^2([-1,1], \mathrm{d}y/2)$, $k \geq 0$. Define also the product measure on U

$$\mathrm{d}\mu(\boldsymbol{y}) := \bigotimes_{j \geq 1} \frac{\mathrm{d}y_j}{2}.$$

We define the Legendre coefficients of $q(\boldsymbol{y})$ by

$$q_{\boldsymbol{\nu}} := \int_U q(\boldsymbol{y}) L_{\boldsymbol{\nu}}(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y}), \quad \forall \boldsymbol{\nu} \in \mathcal{F}.$$

Since $(L_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ is an orthonormal basis,

$$q = \sum_{\boldsymbol{\nu}\in\mathcal{F}} q_{\boldsymbol{\nu}} L_{\boldsymbol{\nu}}$$

with convergence in $L^2(U, V; \mu)$. Suppose that

$$\left\| \frac{\sum_{j\geq 1} \rho_j |\psi_i|}{\overline{u}} \right\|_{L^\infty(D)} < 1 \; .$$

(4.1h) For $\boldsymbol{y}, \boldsymbol{z} \in U$, define $T_{\boldsymbol{y}}\boldsymbol{z} := (y_j + (1 - |y_j|)\rho_j z_j)_{j\geq 1}$. Furthermore, for $\boldsymbol{y}, \boldsymbol{z} \in U$, $\bar{u}_{\boldsymbol{y}} := \bar{u} + \sum_{j\geq 1} y_j \psi_j$, $\psi_{\boldsymbol{y},j} := (1 - |y_j|)\rho_j \psi_j$, $j \geq 1$. Let $q_{\boldsymbol{y}}$ be the solution of

$$\begin{cases} -\mathrm{div}\Big(\Big(\bar{u}_{\boldsymbol{y}} + \sum_{j\geq 1} z_j \psi_{\boldsymbol{y},j}\Big) \nabla q_{\boldsymbol{y}}(\boldsymbol{z})\Big) & = f, \\ q(\boldsymbol{y})\Big|_{\partial D} = 0. \end{cases}$$

Define the Taylor coefficients $t_{\boldsymbol{y},\boldsymbol{\nu}} := (\boldsymbol{\nu}!)^{-1}(\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} q_{\boldsymbol{y}}(\boldsymbol{z}))|_{\boldsymbol{z}=0}$, $\boldsymbol{\nu} \in \mathcal{F}$. Prove that there exists a constant $C > 0$ such that for every $\boldsymbol{y} \in U$

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|t_{\boldsymbol{y},\boldsymbol{\nu}}\|_V^2 \leq C\|f\|_{V'}^2.$$

HINT: Apply the result of (4.1d).

(4.1i) Prove that there exists a constant $C > 0$ such that

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \prod_{j\geq 1} (2\nu_j + 1)^{-1} \boldsymbol{\rho}^{2\boldsymbol{\nu}} \|q_\nu\|_V^2 \leq C\|f\|_{V'}^2.$$

HINT: You may use Rodrigues' formula:

$$L_{\boldsymbol{\nu}}(\boldsymbol{y}) = \prod_{j\geq 1} \partial_{y_j}^{\nu_j} \left(\frac{\sqrt{2\nu_j + 1}}{\nu_j! 2^{\nu_j}} (y_j^2 - 1)^{\nu_j} \right), \quad \forall \boldsymbol{\nu} \in \mathcal{F}, \forall \boldsymbol{y} \in U.$$

(4.1j) Let $p \in (0, 2)$ and define $p' := 2p/(2-p)$. Finally prove using the property $\boldsymbol{\rho}^{-1} \in \ell^{p'}(\mathbb{N})$ that $(\|q_{\boldsymbol{\nu}}\|_V)_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$.

Part 3:

Suppose that the Dirichlet Laplacean is boundedly invertible from $H^2(D) \cap V$ onto $L^2(D)$, i.e.,

$$(-\Delta)^{-1} : L^2(D) \to H^2(D) \cap V$$

is bounded. This is for example true if D has a smooth boundary or D is a convex polygon. Assume that $f \in L^2(D)$. Suppose that for $\boldsymbol{\rho} \in (1, \infty)^{\mathbb{N}}$,

$$\left\| \frac{\sum_{j\geq 1} \rho_j |\psi_i|}{\overline{u}} \right\|_{L^\infty(D)} < 1 \quad \text{and} \quad \left\| |\nabla \bar{u}| + \sum_{j\geq 1} \rho_j |\nabla \psi_j| \right\|_{L^\infty(D)} < \infty.$$

(4.1k) Prove that $\sup_{\boldsymbol{y}\in U} \|q(\boldsymbol{y})\|_{H^2(D)} < \infty$.

(4.1l) Formally rewrite $-\mathrm{div}(u(\boldsymbol{y})\nabla q(\boldsymbol{y})) = -u(\boldsymbol{y})\Delta q(\boldsymbol{y}) - \nabla u(\boldsymbol{y}) \cdot \nabla q(\boldsymbol{y})$. Prove that there exists a constant $C > 0$ such that

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{2\boldsymbol{\nu}} \|t_{\boldsymbol{\nu}}\|^2_{H^2(D)} \leq C\|f\|^2_{L^2(D)}.$$

HINT: Apply the Young inequality, i.e., for every $a, b, \varepsilon \in (0,\infty)$, $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$.

(4.1m) Assume the notation from (4.1h). Prove that there exists a constant $C > 0$ such that for every $\boldsymbol{y} \in U$

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|t_{\boldsymbol{y},\boldsymbol{\nu}}\|^2_{H^2(D)} \leq C\|f\|^2_{L^2(D)}.$$

(4.1n) Prove that there exists a constant $C > 0$ such that

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \prod_{j\geq 1} (2\nu_j + 1)^{-1} \boldsymbol{\rho}^{2\boldsymbol{\nu}} \|q_\nu\|^2_{H^2(D)} \leq C\|f\|^2_{L^2(D)}.$$

(4.1o) Let $p \in (0, 2)$ and define $p' := 2p/(2-p)$. Finally prove using the property $\boldsymbol{\rho}^{-1} \in \ell^{p'}(\mathbb{N})$ that $(\|t_{\boldsymbol{\nu}}\|_{H^2(D)})_{\boldsymbol{\nu}\in\mathcal{F}}, (\|q_{\boldsymbol{\nu}}\|_{H^2(D)})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$.

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