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Numerical Analysis of High-Dimensional Problems

ETH Zürich
D-MATH

Exercise 5

Problem 5.1 Real variable approach

Let X, Y be two Banach spaces. Let $A_0 \in L(X, Y')$ be bounded and bijective and $A_j \in L(X, Y')$ bounded for all $j \in \mathbb{N}$. Set $A(\boldsymbol{y}) = A_0 + \sum_{j\in\mathbb{N}} y_j A_j$ and assume with $B_j := A_0^{-1} A_j$ and $\beta_j := \left\|A_0^{-1} A_j\right\|_{L(X,X)}$ that $\|(\beta_j)_{j\in\mathbb{N}}\|_{\ell^1} < 1$. as well as $\|(\beta_j)_{j\in\mathbb{N}}\|_{\ell^p} < \infty$ for some fixed $p \in (0, 1)$.

(5.1a) Let $k \in \mathbb{N}$ and assume that $q : [-1, 1] \to X$ is differentiable. Show that for $B \in L(X, Y')$ bounded

$$\frac{d^k}{dy^k}(yBq(y)) = yBq^{(k)}(y) + kBq^{(k-1)}(y). \tag{5.1.1}$$

(5.1b) Let $f : U := [-1, 1]^{\mathbb{N}} \to Y'$. Show that for every $\boldsymbol{y} \in U$ there exists a unique $q(\boldsymbol{y}) \in X$ with $A(\boldsymbol{y})q(\boldsymbol{y}) = f(\boldsymbol{y})$.

(5.1c) With the notation $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} = \frac{\partial^{|\boldsymbol{\nu}|}}{\partial y_1^{\nu_1} \partial y_2^{\nu_2} \dots}$ assume that $\left\|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} f)(\mathbf{0})\right\|_{Y'} \le C_f |\boldsymbol{\nu}|! \boldsymbol{\beta}_f^{\boldsymbol{\nu}}$, where $\boldsymbol{\beta}_f = (\beta_{f;j})_{j\in\mathbb{N}} \in \ell^p$. Show that there exists a sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}} \in \ell^p$ and a constant $C < \infty$ such that

$$\left\|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\mathbf{0})\right\|_X \le C|\boldsymbol{\nu}|! \boldsymbol{\gamma}^{\boldsymbol{\nu}} \tag{5.1.2}$$

for all $\boldsymbol{\nu} \in \mathcal{F}$, where

$$\mathcal{F} = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} \,\middle|\, \sum_j \nu_j < \infty \right\} \tag{5.1.3}$$

denotes the set of all finitely supported multiindices.

HINT: Proceed as in the proof of Thm. 2.26 of the lecture notes.

Problem 5.2 Complex variable approach

For $\rho > 0$ denote $B_\rho(z) := \{x \in \mathbb{C} \,|\, |x| \le \rho\}$ and let X be a complex Banach space. In the following, if we say that a mapping is holomorphic on $B_\rho(z)$, we mean that it is holomorphic on some open superset of $B_\rho(z)$[1].

[1] by which we mean complex Fréchet differentiable at every ζ in this superset, i.e. for some $q'(\zeta) \in L(\mathbb{C}, X)$ it holds $|q(\zeta + h) - q(\zeta) - q'(z)(h)| = o(|h|)$ as $h \to 0$ for $h \in \mathbb{C}$. In this situation it is common to identify $q'(\zeta) \in L(\mathbb{C}, X)$ with $q'(\zeta)(1) \in X$, thus we also write $q'(\zeta) \in X$.

(5.2a) Show the Cauchy integral formula in Banach spaces: If $q : B_\rho(z) \to X$ is holomorphic, then

$$q(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_\rho} \frac{q(\zeta)}{\zeta - z} \, \mathrm{d}\zeta. \tag{5.2.1}$$

HINT: Use the Cauchy integral formula for functions mapping from $B_\rho(z) \to \mathbb{C}$, the fact that $\mathbf{x} = \mathbf{y} \in X$ if and only if $f(\mathbf{x}) = f(\mathbf{y})$ for all $f \in X'$, and the fact that $f(\int_{\partial B_\rho(z)} q(z) \, \mathrm{d}z) = \int_{\partial B_\rho(z)} f(q(z)) \, \mathrm{d}z$ (assume these statements as given, there's no need to prove them).

(5.2b) Let $q : B_\rho \to X$ be a holomorphic function with $\sup_{\zeta \in B_\rho} \|q(\zeta)\|_X \le C < \infty$. Show that for every $k \in \mathbb{N}$

$$\left\| \frac{d^k}{dz^k} q(z) \Big|_{z=0} \right\|_X \le C \frac{k!}{\rho^k}. \tag{5.2.2}$$

HINT: Use the Cauchy integral formula.

(5.2c) Let $k \in \mathbb{N}$ and let $q : B_{\rho_1}(z_1) \times \cdots \times B_{\rho_k}(z_k) \to X$ holomorphic in all k variables such that $\sup_{\zeta_j \in B_{\rho_j}(z_j) \, \forall j} \|q(\zeta_1, \ldots, \zeta_k)\|_X = C < \infty$, where $\rho_j > 0$ for all $j = 1, \ldots, k$. Prove that for $\boldsymbol{\nu} \in \mathbb{N}_0^k$ it holds

$$\left\| (\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} q)(\boldsymbol{z}) \Big|_{\boldsymbol{z}=\mathbf{0}} \right\|_X \le C \frac{\boldsymbol{\nu}!}{\boldsymbol{\rho}^{\boldsymbol{\nu}}}. \tag{5.2.3}$$

(5.2d) Let $q : [-1, 1]^{\mathbb{N}}$ satisfy the following: For some fixed $p \in (0, 1)$ there exists a sequence $(b_j)_{j \in \mathbb{N}} \in \ell^p$ and $\varepsilon > 0$ such that

i) For every sequence $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}} \in (1, \infty)^{\mathbb{N}}$ of numbers satisfying

$$\sum_{j \in \mathbb{N}} (\rho_j - 1) b_j < \varepsilon, \tag{5.2.4}$$

the map q allows an extension $\tilde{q} : B_{\boldsymbol{\rho}} := B_{\rho_1}(0) \times B_{\rho_2}(0) \times \cdots \to X$ that is holomorphic as a function of each variable.

ii) There exists $B_0 < \infty$ such that for every extension $\tilde{q}$ in i) it holds $\sup_{\boldsymbol{z} \in B_{\boldsymbol{\rho}}} \|\tilde{q}(\boldsymbol{z})\|_X \le B_0$.

Show that there exists a constant $C < \infty$ and a sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \in \mathbb{N}} \in \ell^p$ with $\|\boldsymbol{\gamma}\|_{\ell^\infty} < 1$ such that for all $\boldsymbol{\nu} \in \mathcal{F}$

$$\|(\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} q)(\mathbf{0})\|_X \le C |\boldsymbol{\nu}|! \boldsymbol{\gamma}^{\boldsymbol{\nu}}. \tag{5.2.5}$$

HINT: Proceed as in the proof of Thm. 2.36 of the lecture notes. Choose $\boldsymbol{\rho}$ in subproblem (5.2c) depending on $\boldsymbol{\nu}$.

Problem 5.3 Affine parametric problem with complex approach

Assuming f in (5.1c) to be independent of $\boldsymbol{y}$, prove the statement of (5.1c) using (5.2d).

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