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Numerical Analysis of High-Dimensional Problems

ETH Zürich
D-MATH

Exercise 6

Problem 6.1 Analyticity on Ellipse

(6.1a) If

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{z-t}\mathrm{d}t, \quad n \in \mathbb{N}_0,$$

and $w = \frac{1}{2}(z + z^{-1})$, then

$$Q_n(w) = \sum_{k=n+1}^{\infty} \frac{\sigma_{nk}}{z^k},$$

where

$$|\sigma_{nk}| \le \pi, \quad n \in \mathbb{N}_0, k = n+1, n+2, \dots.$$

HINT: Use $\sigma_{nk} = \int_0^\pi P_n(\cos(\theta))\sin(k\theta)\mathrm{d}\theta$.

(6.1b) Let $f(z)$ be analytic in the interior of $\mathcal{E}_\rho$, $\rho > 1$, but not analytic in the interior of any $\mathcal{E}_{\tilde{\rho}}$, $\tilde{\rho} > \rho > 1$. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$$

with

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\mathrm{d}x.$$

Show that

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{\rho}.$$

Problem 6.2 Gevrey functions

We say that $g \in C^\infty([-1,1])$ is Gevrey regular of index $s \ge 1$ in $(-1,1)$ if there exists a constant $C_g > 0$ such that

$$\forall k \in \mathbb{N}_0 : \quad \|f^{(k)}\|_{L^\infty(-1,1)} \le C_g^{k+1}(k!)^s.$$

The set of all functions which are Gevrey regular of index $s \ge 1$ is denoted by $\mathcal{G}^s(-1,1)$.

(6.2a) Show that all $g \in \mathcal{G}^1(-1,1)$ are analytic on $[-1,1]$.

HINT: Cover $[-1,1]$ with finitely many balls of radius $0 < r < C_g^{-1}$ and consider in each ball a Taylor series of g in their center.

(6.2b) Show that all $g \in \mathcal{G}^1(-1,1)$ are analytic on $\mathcal{E}_\rho \subset \mathbb{C}$ for some $\rho > 0$. Which is the largest $\rho > 0$ such that you can verify this property?

(6.2c) Consider

$$f(x) := \begin{cases} \exp(-x^{-1}) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that $f \in \mathcal{G}^3(-1,1)$.

HINT: Use the Faa di Bruno Formula in the form:

$$(f \circ g)^{(n)}(x) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(x)) \prod_{B \in \pi} g^{|B|}(x), \tag{6.2.1}$$

where Π denotes the set of all 2^n partitions of $\{1, \ldots, n\}$.

REMARK: In fact it holds $f \in \mathcal{G}^2(-1,1)$, prove it if you can.

(6.2d) Show that f is not analytic at $x = 0$.

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