Mathematical Finance

Exercise sheet 1

Exercise 1.1 Let X_i be independent Bernoulli random variables with $P(X_i = +1) = \frac{1+\alpha_i}{2}$ and $P(X_i = -1) = \frac{1-\alpha_i}{2}$, and $0 < \alpha_i < 1$, $i \in \mathbb{N}$.

(a) Let $\alpha_i = 2/3$ and consider a one-period model with a countable number of assets whose prices are given by $S_0^i = 1$, $S_1^i = 1 + X_i$, $i \in \mathbb{N}$. Does there exist a probability measure Q equivalent to P such that $E_Q[S_1^i] = S_0^i$ for every $i \in \mathbb{N}$?

(b) Consider an infinite horizon model with one stock $S = (S_n)_{n\in\mathbb{N}}$ whose price is given by $S_0 = 1$, $S_n = S_{n-1} + X_n$, $n \in \mathbb{N}$. Show that there exists a probability measure Q equivalent to P such that $E_Q[S_{n+1} \mid \mathcal{F}_n] = S_n$ for every $n \in \mathbb{N}$ if and only if $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ be a filtration on Ω. If X is a real-valued $\mathbb{F}$-adapted process and B is a Borel subset of $\mathbb{R}$, then

$$\inf\{t \in I : X_t \in B\}$$

is called the *hitting time* of X on B.

(a) Let $I = \mathbb{N}$. Show every stopping time is a hitting time, and every hitting time is a stopping time.

(b) Let $I = \mathbb{R}_+$. Show every stopping time is a hitting time. Assume that the filtration is the natural filtration of a càdlàg X and give an example of a hitting time of X that is not a stopping time.

Exercise 1.3 **(Python)** Let B be a Brownian motion and $\tau = \inf\{t \in \mathbb{R}_+ : B_t = 1\}$.

(a) Plot the distribution of B_1 and compute the mean $E[B_1]$.

(b) Plot the distribution of B_τ and compute the mean $E[B_\tau]$.