Mathematical Finance

Exercise sheet 2

Exercise 2.1 Let X and Y be random variables with $E[X \mid Y] \overset{a.s.}{=} Y$ and $E[Y \mid X] \overset{a.s.}{=} X$.

(a) Assume that X and Y are square-integrable and prove that $X \overset{a.s.}{=} Y$.

(b) Assume that X and Y are integrable and prove that $X \overset{a.s.}{=} Y$.

Exercise 2.2 Let $\mathbb{S}$ denote the family of simple predictable processes, i.e.,

$$H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{]\tau_i, \tau_{i+1}]}$$

for stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_{i+1} < \infty$ and bounded $\mathcal{F}_{\tau_i}$-measurable H_i for $i = 0, 1, \ldots, n+1$, $\mathbb{D}$ denote the family of adapted càdlàg processes and $\mathbb{L}$ denote the family of adapted càglàd processes on $[0, \infty[$. We endow $\mathbb{D}$ and $\mathbb{L}$ with the topology of convergence uniformly on compacts in probability generated by the metric

$$d(X, Y) := \sum_{k=1}^{\infty} \mathbb{E}[2^{-k} \wedge \sup_{s \leqslant k} |X_s - Y_s|].$$

Moreover, let the space of measurable random variables L^0 be endowed with the topology generated by the convergence in probability. Show that

(a) The vector spaces $\mathbb{L}$ and $\mathbb{D}$ are complete.

(b) The continuity of $J_X : \mathbb{S} \to \mathbb{D}$ with $J_X(H) := H_0 X_0 + \sum_{i=1}^{n} H_i (X_{\tau_{i+1} \wedge t} - X_{\tau_i \wedge t})$, for $H \in \mathbb{S}$, in the u.c.p. metric on $\mathbb{S}$, i.e., that X is a good integrator, is equivalent to that, for every fixed $t \in [0, \infty[$, the mapping $I_{X^t} : \mathbb{S} \to L^0$ with $I_{X^t}(H) := J_X(H)_t$, for $H \in \mathbb{S}$, is continuous in the uniform norm metric on $\mathbb{S}$.

Exercise 2.3 (Python) Let B be a Brownian motion modeling a stock.

(a) Compute the expected value of an European put option $(K - B_t)^+$ with the strike price $K = 1$ at the maturity $t = 100$.

In (b) and (c), we consider the following trading strategy. Start with one stock, sell it if the price reaches 2, after this, buy a new stock if the price falls below 1 after which sell the stock if the price reaches 2, and keep repeating this procedure.

(b) Compute the expected number of times one has sold a stock prior to $t = 100$.

(c) Plot the distribution and compute the expected value of the portfolio at $t = 100$.