Mathematical Finance

Exercise sheet 3

Exercise 3.1

(a) Let X be an adapted càdlàg process. Show that X is a good integrator if and only if X is a local good integrator.

(b) Let B be a standard Brownian motion and $\tau = \inf\{t \geqslant 0 : B_t = 1\}$. Put

$$Y_t = \begin{cases} B^{\tau}_{t/(1-t)} & \text{if } t < 1, \\ 1 & \text{if } t \geqslant 1. \end{cases}$$

Assume the (augmented) natural filtration and show that the process Y is a local martingale, but not a martingale.

Exercise 3.2

(a) Let x be a càdlàg function on $[0,1]$, and let π^n be a refining sequence of dyadic rational partitions of $[0,1]$ with $\lim_{n\to\infty} \operatorname{mesh}(\pi^n) = 0$. Show that, if the sum

$$\sum_{t_k^n, t_{k+1}^n \in \pi^n} y(t_k^n)(x(t_{k+1}^n) - x(t_k^n))$$

converges to a finite limit for every càglàd function on $[0,1]$, then x is of finite variation.

(b) Let X be a good integrator, and let Π^n be a sequence of partition tending to identity. Show that

$$\sum_{\tau_k^n, \tau_{k+1}^n \in \Pi^n} Y_{\tau_k^n}(X_{\tau_{k+1}^n} - X_{\tau_k^n}) \stackrel{u.c.p.}{\to} (Y \bullet X)$$

for every adapted càglàd process Y.

Exercise 3.3 Let X be a good integrator with $X_0 = 0$. Show that the process

$$Z_t = \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \Pi_{s \leqslant t}(1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right), \ t \in [0, \infty[,$$

solves the SDE

$$dZ = Z_- dX, \ Z_0 = 1.$$

Exercise 3.4 (Python)

(a) Implement a function to simulate paths of a Poisson process.

(b) Plot paths of a compensated Poisson process with different values of the intensity parameter.