Mathematical Finance

Exercise sheet 4

Exercise 4.1 Let X be a good integrator and A a finite variation process. Show that their covariation process is

$$[X, A]_t = \sum_{s \leqslant t} \Delta X_s \Delta A_s.$$

Exercise 4.2 Let X be a continuous local martingale vanishing at $t = 0$. Show that X is a Brownian motion if and only if $[X, X]_t = t$.

Exercise 4.3 Let (π^n) be a refining sequence of deterministic partitions tending to identity and assume that M is a càdlàg process such that, for every n, a simple sampling process M^{π^n} is a martingale and

$$\sup_n E[|M_1^{\pi^n}|^2] < \infty.$$

Show that there exists a sequence of refining deterministic partitions (π^{n_k}) such that $M_1^{\pi^{n_k}}$ converges in the weak topology to a random variable ξ with

$$M_t = E[\xi \mid \mathcal{F}_t], \; t \in [0, 1[.$$

Exercise 4.4 (Python) Compute numerically the quadratic variation at some time $t > 0$ for

(a) a Brownian motion $B = (B_t)_{t \geqslant 0}$,

(b) a Poisson process $N^\lambda = (N_t^\lambda)_{t \geqslant 0}$ for some $\lambda > 0$,

(c) a compensated Poisson process $\widetilde{N}^\lambda = (N_t^\lambda - \lambda t)_{t \geqslant 0}$ for some $\lambda > 0$.