Mathematical Finance

Exercise sheet 6

Exercise 6.1 Let X_k be independent Bernoulli random variables with $P(X_k = +1) = P(X_k = -1) = \frac{1}{2}$, $k \in \mathbb{N}$. Consider an infinite horizon model with a constant bank account normalized to one and a stock $S = (S_k)_{k\in\mathbb{N}}$ whose price is given by $S_0 = 1$, $S_k = S_{k-1} + X_k,\ k \in \mathbb{N}$. Consider the following strategy. Start with zero initial wealth, buy one stock and keep doubling your stock holdings until the stock goes up for the first time, then sell the stocks.

(a) Find the self-financing strategy $\varphi = (\eta, \theta)$ and the associated wealth process $V = (V_k(\varphi))_{k\in\mathbb{N}}$ with zero initial wealth for this strategy.

(b) Show that with this strategy, $V_\infty(\varphi) := \lim_{k\to\infty} V_k(\varphi) = 1$ a.s..

(c) Put

$$Y_t := \begin{cases} V_k(\varphi) & \text{for } 1 - \frac{1}{k+1} \leqslant t < 1 - \frac{1}{k+2}, \\ 1 & \text{for } t \geqslant 1. \end{cases}$$

Assume the (augmented) natural filtration and show that the process Y is a local martingale, but not a martingale.

Exercise 6.2 Show that every local martingale is locally integrable.

Exercise 6.3 Recall that a process X which can be decomposed into a local martingale M and a predictable FV process A is called a special semimartingale. Show that, for every special semimartingale X, choosing $A_0 = 0$, the decomposition $X = M + A$ is unique.

Exercise 6.4 (Python) Let $(B_t)_{t\geqslant 0} = (B_t^1, B_t^2, B_t^3)_{t\geqslant 0}$ be a standard three-dimensional Brownian motion starting at $B_0 = (1, 0, 0)$. Set

$$M_t := ||B_t||^{-1},\ t \geqslant 0,$$

and

$$\tau := \inf\{t \geqslant 0 : M_t = 2\}.$$

(a) Simulate the paths of $M = (M_t)_{t\geqslant 0}$.

(b) Compute the expectation of M_t at $t = 2$.

(c) Compute the expectation of M_t^τ at $t = 2$.