# Mathematical Finance

## Exercise sheet 9

**Exercise 9.1** Let $L$ be a Lévy process. Show that

(a) The process $\exp(L)$ is a martingale, if it is a sigma-martingale.

(b) The process $L$ is a martingale, if it is a sigma-martingale.

**Exercise 9.2** Let $w \in C^2([0,\infty])$. Assume that the law of $S_T$ admits a density and show that

$$E[w(S_T)] = w(S_0) + \int_0^{S_0} w''(K)P(K)dK + \int_{S_0}^{\infty} w''(K)C(K)dK,$$

where $S_0 = E[S_T]$, $C(K) = E[(S_T - K)^+]$ and $P(K) = E[(K - S_T)^+]$.

**Exercise 9.3** Let

$$V_t = \text{ess sup}_{t\leqslant\tau\leqslant T} E[U_\tau \mid \mathcal{F}_t],\ t \in [0,T], \tag{1}$$

with $V_0 < \infty$ and $U$ càdlàg with $E[\sup_{0\leqslant t\leqslant T} |U_t|^p] < \infty$ for some $p > 1$. Show that

$$V_0 = \inf_{M\in\mathcal{H}_0^1} E[\sup_{0\leqslant t\leqslant T}(U_t - M_t)], \tag{2}$$

where $\mathcal{H}_0^1 := \{M \in \mathcal{H}^1 : M_0 = 0\}$.

**Exercise 9.4 (Python)** Assume Black-Scholes dynamics for $S$, say $S_0 = 100$ and $(r,\mu,\sigma,T) = (0.06, 0.0, 0.4, 0.5)$, and that $U$ in (1) represents the discounted exercise value for the put option on $S$ with the strike price $K = 100$. Compute an upper bound for $V_0$ by choosing $M$ in (2) to be the (centered) value process of the corresponding European put option on $S$ with the strike price $K$ and the maturity $T$.