

# Mathematical Finance

## Exercise sheet 10

### Exercise 10.1

- (a) Give an example of a process with independent increments that is not a semimartingale.
- (b) Give an example of a process with stationary increments that is not a semimartingale.

### Exercise 10.2

 Assume

$$dS_t = \sigma(t, S_t)S_t dW_t, \quad S_0 = s_0 > 0,$$

for  $C^{1,2}$ -function  $\sigma$  and assume that there exists  $C^{1,2}$ -function  $f$  such that  $f(t, \cdot)$  is the density for  $S_t$  for all  $t \geq 0$ . Show that

$$\sigma(T, K) = \frac{1}{K} \sqrt{\frac{2 \frac{\partial C}{\partial T}(T, K)}{\frac{\partial^2 C}{\partial K^2}(T, K)}}.$$

Hint: Consider the value process for some  $h(S_T)$ .

### Exercise 10.3

 Let

$$dY_t = \kappa(\theta - Y_t)dt + \beta\sqrt{Y_t}dW_t, \quad Y_0 = y_0 > 0, \tag{1}$$

where  $W$  is a  $Q$ -Brownian motion,  $\kappa, \theta, \beta$  are constants satisfying the Feller condition  $2\kappa\theta > \beta^2$ . Show that

$$E_Q \left[ \frac{1}{T} \int_0^T Y_t dt \right] = \frac{1 - e^{-\kappa T}}{\kappa T} Y_0 + \left( 1 - \frac{1 - e^{-\kappa T}}{\kappa T} \right) \theta. \tag{2}$$

**Exercise 10.4 (Python)** Compute the expectation (2) by simulating the paths of (1).