Mathematical Finance

Exercise sheet 12

Exercise 12.1 Let the financial market $S = (S_t)_{t=0}^T$ be defined over the *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ and satisfy $\mathcal{M}^a(S) \neq \varnothing$, and let U be a utility function satisfying the Inada conditions. Consider the value functions

$$u(x) = \sup_{X_T \in C(x)} E[U(X_T)] \text{ and } v(y) = \inf_{Q \in \mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where V is the convex conjugate of U and $C(x) = \{X_T \in L^0(\Omega, \mathcal{F}_T, P) : E_Q[X_T] \leqslant x \ \forall Q \in \mathcal{M}^a(S)\}$. Show that the optimizers $\widehat{X}_T(x)$, $\widehat{Q}(x)$ and $\widehat{y}(x)$ satisfy $U'\left(\widehat{X}_T(x)\right) = \widehat{y}(x)\frac{d\widehat{Q}(x)}{dP}$ for each $x \in \mathrm{dom}(U)$.

Exercise 12.2 Assume a Black-Scholes model on which the wealth process X evolves according to

$$dX_t = \alpha_t\left((\mu - r)dt + \sigma dW_t\right), \quad X_0 = x, \tag{1}$$

where the process α describing the amount of money invested in the stock at time $t \in [0, T]$ is in $\mathcal{A}$; the set of all progressively measurable processes taking values in a closed convex set $A \subseteq \mathbb{R}$ such that the SDE (1) admits a unique strong solution. The goal is to maximize the expected utility from the terminal wealth at the terminal time $T < \infty$ subject to some non-decreasing and concave utility function U.

(a) Show that for all $t \in [0, T]$, the value function $v(t, \cdot)$ is concave in x.

(b) Write down the dynamic programming principle and the Hamilton-Jacobi-Bellman equation for this stochastic control problem and specify the terminal condition.

Exercise 12.3 Assume that the utility function in the previous exercise is $U(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$, for some $\gamma > 0$ and solve the utility maximization problem.

Exercise 12.4 (Python) Plot the paths of the wealth process for different values of the risk aversion parameter γ.