Mathematical Finance

Exercise sheet 14

Exercise 14.1 Let S be a semimartingale satisfying NFLVR. Assume that S is complete in the sense that there exists a unique equivalent σ-martingale measure Q for S. Let $U : (0,\infty) \to \mathbb{R}$ be a utility function satisfying Inada conditions and such that $u(x) < \infty$ for some (and hence all) $x \in (0,\infty)$. For $x > 0$ and $y > 0$, define the sets

$$\mathcal{C}(x) := \{X_T \in L^0_+(\Omega, \mathcal{F}, P) : X_T \leqslant x + (H \cdot S)_T \text{ for admissible } H\}$$
$$\mathcal{D}(y) := y\mathcal{D}(1) := y\{Y_T \in L^0_+(\Omega, \mathcal{F}, P) : E[X_T Y_T] \leqslant 1 \ \forall X_T \in \mathcal{C}(1)\}.$$

(a) Fix $y > 0$. Show that

$$Y_T \leqslant y\frac{dQ}{dP} \ \ P\text{-a.s. for all } Y_T \in \mathcal{D}(y),$$

where $\frac{dQ}{dP}$ denotes the density of Q with respect to P on $\mathcal{F}_T$. Deduce that

$$v(y) = \inf_{Y_T \in \mathcal{D}(y)} E[V(Y_T)] = E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where $E[V(Y_T)] := +\infty$ if $V^+(Y_T) \notin L^1(P)$.

(b) Let $y_0 := \inf\{y > 0 : v(y) < \infty\}$. Show that the function v is in $C^1(y_0,\infty)$ and satisfies

$$v'(y) = E\left[\frac{dQ}{dP}V'\left(y\frac{dQ}{dP}\right)\right], \quad y \in (y_0,\infty).$$

(c) Set $x_0 := \lim_{y\downarrow\downarrow y_0} -v'(y)$. Fix $x \in (0, x_0)$. Let $\widehat{y}(x) \in (y_0,\infty)$ be the unique number such that $-v'(\widehat{y}(x)) = x$. Show that $\widehat{X}_T := (U')^{-1}\left(\widehat{y}(x)\frac{dQ}{dP}\right)$ is the unique solution to the primal problem

$$u(x) = \sup_{X_T \in \mathcal{C}(x)} E[U(X_T)].$$

Exercise 14.2 Give an example of the situation on which there exists a dual optimizer $\operatorname{argmin}_{Y_T \in \mathcal{D}(1)} E[V(Y_T)]$ in $\mathcal{D}(1)$, but it is not of the form $\frac{dQ}{dP}$ for any EσMM Q.

Exercise 14.3 Let L^0_+ denote the collection of all nonnegative random variables and L^0_{++} be the family of all strictly positive random variables (see Exercise Sheet 3 for the relevant definitions). Assume that $\mathcal{C} \subset L^0_+$ is a convex set satisfying $L^0_{++} \cap \mathcal{C} \neq \varnothing$ and closed in probability.

(a) Show that if there exists an $\hat{f} \in L^0_{++} \cap \mathcal{C}$ such that $E[f/\hat{f}] \leqslant 1$ for all $f \in \mathcal{C}$, then $\mathcal{C}$ is bounded in probability, i.e., $\limsup_{M\to\infty} \sup_{f\in\mathcal{C}} P[f \geqslant M] = 0$.

(b) Now we assume that $\mathcal{C}$ is bounded in probability. Show that its solid hull

$$\mathcal{C}' := \{f \in L^0_+ | f \leqslant h \text{ for some } h \in \mathcal{C}\}$$

is also closed in probability, convex and bounded in probability.

(c) Show that if $\mathcal{C} = \mathcal{C}'$ and $1 \in \mathcal{C}$ (here 1 is the constant random variable $X = 1$), then for all $n \geqslant 1$, the set $\mathcal{C}^n := \{f \in \mathcal{C} | f \leqslant n\}$ is nonempty and the utility maximization problem $\sup_{f\in\mathcal{C}^n} E[\log f]$ has a maximizer f^n satisfying $E[\log f] = \sup_{f\in\mathcal{C}} E[\log f] \in (-\infty, +\infty)$.

(d) Keeping all assumptions from **c)** above and still denote by f^n a maximizer for $\sup_{f\in\mathcal{C}^n} E[\log f]$. Define for each $\epsilon \in (0, \frac{1}{2}]$ and $f \in \mathcal{C}^n$ a quantity

$$\Delta_\epsilon(f|f^n) := \frac{\log\big((1-\epsilon)f^n + \epsilon f\big) - \log f^n}{\epsilon}.$$

Use the elementary inequality $\log y - \log x \leqslant \frac{y-x}{x}$ for all $0 < x < y$ to show that $\Delta_\epsilon(f|f^n) \geqslant -2$ for all $\epsilon \in (0, \frac{1}{2}]$ and for all $f \in \mathcal{C}^n$. Then show that $E[\frac{f}{f^n}] \leqslant 1$ for all $f \in \mathcal{C}^n$.

(e) Now for each n we have a maximizer f^n for $\sup_{f\in\mathcal{C}^n} E[\log f]$. Apply the Komlos Lemma for the sequence (f^n) we get an $\hat{f} \in \mathcal{C}$. Show that this $\hat{f}$ satisfies that $1/\hat{f} \in L^0_{++}$ and $E[\frac{f}{\hat{f}}] \leqslant 1$ for all $f \in \mathcal{C}$.

(f) Now we assume neither $1 \in \mathcal{C}$ nor $\mathcal{C} = \mathcal{C}'$, but still assume that $\mathcal{C}$ is bounded in probability. Show that in this case we still have an $\hat{f} \in \mathcal{C}$ such that $\frac{1}{\hat{f}} \in L^0_{++}$ and $E[\frac{f}{\hat{f}}] \leqslant 1$ for all $f \in \mathcal{C}$. We call $\hat{f}$ the *static numéraire* in $\mathcal{C}$.

Exercise 14.4 Let $\mathcal{X}$ be a set of semimartingales defined on a finite time horizon $[0, T]$. We call $\mathcal{X}$ a *wealth process set* if it satisfies all the following conditions:

1. Each $X \in \mathcal{X}$ is nonnegative, adapted, RCLL and $X_0 = 1$.
2. There exists a strictly positive process $\overline{X} \in \mathcal{X}$.
3. $\mathcal{X}$ is convex, i.e., for $X^1, X^2 \in \mathcal{X}$ and $\lambda \in [0,1]$, $(1-\lambda)X^1 + \lambda X^2 \in \mathcal{X}$.
4. $\mathcal{X}$ is *decomposable*, that is, for all $t \in [0, T]$ and all $A \in \mathcal{F}_t$, for all $X \in \mathcal{X}$ and all strictly positive $X' \in \mathcal{X}$, the process

$$1_{A^c} X_s + 1_A \frac{X'_{t\vee s}}{X'_t} X_{t\wedge s} = \begin{cases} X_s(\omega), \text{ if } s \leqslant t, \text{ or } \omega \in A^c; \\ \frac{X_t(\omega)}{X'_t(\omega)} X'_s(\omega), \text{ if } s > t \text{ and } \omega \in A \end{cases}$$

is also an element of $\mathcal{X}$.

(a) Give an economic interpretation of the decomposability in words.

(b) Let $\mathcal{X}^1 = 1 + G(\Theta^1_{adm}(S))$. Show that $\mathcal{X}^1$ is a wealth process set.

(c) We call a wealth process set $\mathcal{X}$ satisfies the NUPBR condition if the set $\mathcal{X}_T := \{X_T | X \in \mathcal{X}\}$ is bounded in probability. Show that if a wealth process set $\mathcal{X}$ satisfies NUPBR and $\mathcal{X}_T$ is closed in probability, then there exists an $\widehat{X} \in \mathcal{X}$ such that $\frac{1}{\widehat{X}}$ is strictly positive and $\frac{X}{\widehat{X}}$ is a supermartingale for all $X \in \mathcal{X}$.