Mathematical Finance

Solution sheet 1

Solution 1.1

(a) The answer is negative. Indeed, assume for a contradiction that there exists a measure Q equivalent to P such that $E_Q[X_i] = 0$ for every $i \in \mathbb{N}$. By the law of large numbers, $Y_n/n \overset{P-a.s.}{\to} 1/3$, where $Y_n = X_1 + \cdots + X_n$. Hence, $Y_n/n \overset{Q-a.s.}{\to} 1/3$. As $|Y_n/n| \leqslant 1$, we get $E_Q[Y_n/n] \to 1/3$, which is a contradiction.

(b) The measure Q must satisfy $Q(X_n = +1) = Q(X_n = -1) = \frac{1}{2}$. For the marginals of P and Q, we have $\frac{dP^n}{dQ^n} = (1+\alpha_n)1_{\{X_n=+1\}} + (1-\alpha_n)1_{\{X_n=-1\}}$ and $\frac{dQ^n}{dP^n} = \frac{1}{1+\alpha_n}1_{\{X_n=+1\}} + \frac{1}{1-\alpha_n}1_{\{X_n=-1\}}$. By Kakutani's dichotomy theorem, we have $P \sim Q$ if and only if

$$\prod_{n=1}^{\infty} \int \left(\frac{dP^n}{dQ^n}\right)^{\frac{1}{2}} dQ^n > 0 \text{ and } \prod_{n=1}^{\infty} \int \left(\frac{dQ^n}{dP^n}\right)^{\frac{1}{2}} dP^n > 0.$$

Both inequalities yield the same condition:

$$\begin{aligned}
& \prod_{n=1}^{\infty}\left(\sqrt{\frac{1}{2}(1+\alpha_n)} + \sqrt{\frac{1}{2}(1-\alpha_n)}\right) > 0 \\
x = \log e^x :\iff & \sum_{n=1}^{\infty} \log\left(\sqrt{\frac{1}{2}(1+\alpha_n)} + \sqrt{\frac{1}{2}(1-\alpha_n)}\right) > -\infty \\
e^{-2x} < 1-x < e^{-x},\ 0 < x < \frac{1}{2} :\iff & \sum_{n=1}^{\infty}\left(1 - \left(\sqrt{\frac{1}{2}(1+\alpha_n)} + \sqrt{\frac{1}{2}(1-\alpha_n)}\right)\right) < \infty \\
\iff & \sum_{n=1}^{\infty} \alpha_n^2 < \infty.
\end{aligned}$$

The last equivalence follows from the fact that $(\sqrt{a}-\sqrt{b})^2 = a - 2\sqrt{ab} + b$ for $a, b \geqslant 0$, and

$$\frac{(x-y)^2}{4(1-\delta)} \leqslant (\sqrt{x}-\sqrt{y})^2 = \left(\int_x^y \frac{dt}{2\sqrt{t}}\right)^2 \leqslant \frac{(x-y)^2}{4\delta}$$

for $0 < \delta < \frac{1}{2}$ and $\delta < x, y < 1-\delta$.

Solution 1.2

(a) If τ is a stopping time, then the process $1_{\{\tau > k\}}$ is adapted to the filtration $\mathbb{F}$. Moreover, $1_{\{\tau>k\}}(\omega)$ is 1 for $\tau(\omega) > k$ and 0 for $\tau(\omega) \leqslant k$ so that

$$\tau(\omega) = \inf\{k \in \mathbb{N} : 1_{\{\tau>k\}}(\omega) \in \{0\}\},$$

i.e., τ is the hitting time of $1_{\{\tau>k\}}$ on the set $\{0\}$. Conversely an arbitrary hitting time of an adapted process X to a Borel set B satisfies

$$\{\tau \leqslant k\} = \bigcup_{j=0}^{k} X_j^{-1}(B) \in \mathcal{F}_k$$

and therefore is a stopping time.

(b) Similarly to (a), given a stopping time τ, we take the process $1_{\{\tau>t\}}$ is adapted to the filtration $\mathbb{F}$. Moreover, $1_{\{\tau>t\}}(\omega)$ is 1 for $\tau(\omega) > t$ and 0 for $\tau(\omega) \leqslant t$ so that

$$\tau(\omega) = \inf\{t \in \mathbb{R}_+ : 1_{\{\tau>k\}}(\omega) \in \{0\}\},$$

i.e., τ is the hitting time of $1_{\{\tau>t\}}$ on $\{0\}$. However, $I = \mathbb{R}_+$ is uncountable and (without further assumptions on the filtration) there exists hitting times that are not stopping times. Consider a càdlàg process X and an open set O, say $O :=]a, \infty[$ for some $a \in \mathbb{R}$. Then, for

$$\tau = \inf\{t \in \mathbb{R}_+ : X_t(\omega) \in O\},$$

we have

$$\{\tau < t\} = \bigcup_{t>q\in\mathbb{Q}_+} \{\tau < q\} \in \mathcal{F}_t,$$

but

$$\{\tau \leqslant t\} = \bigcap_{t<q\in\mathbb{Q}_+} \{\tau < q\} \in \mathcal{F}_{t+}, \tag{1}$$

where $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$. The inclusion $\mathcal{F}_t \subset \mathcal{F}_{t+}$ can be strict without further assumptions, and this constitutes the desired counterexample. Moreover, remark that we assumed the path regularity *for every* $\omega \in \Omega$. In practice, one usually assumes a càdlàg version of the process so that the paths are càdlàg *almost surely*. If so, we have to augment the natural filtration of X so that the event $\{X \text{ is càdlàg}\}$ is measurable at time $t = 0$, then we can use the argument presented above to show that the hitting times of X to open sets are stopping times with respect to a right-continuous filtration. However, a right-continuous filtration is not sufficient to guarantee that a hitting time to an arbitrary Borel set is a stopping time. We have to augment the initial σ-algebra. Let $B \in \mathcal{B}(\mathbb{R}_+)$. We have

$$\tau = \inf\{t \in \mathbb{R}_+ : X_t(\omega) \in B\} = \inf\{t \in \mathbb{R}_+ : (\omega, t) \in X^{-1}(B)\},$$

and, for any $t \in \mathbb{R}_+$,

$$\{\tau < t\} = \pi\left(\{(\Omega \times [0, t[) \cap X^{-1}(B)\}\right),$$

where π denotes the projection from $\Omega \times \mathbb{R}_+$ onto Ω. Since X is càdlàg and adapted, X is progressively measurable and the set $(\Omega \times [0, t[) \cap X^{-1}(B)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. So, by the measurable projection theorem, we have

$$\{\tau < t\} \in \widehat{\mathcal{F}}_t,$$

where $\widehat{\mathcal{F}}_t$ is the universal completion of $\mathcal{F}_t$. Recall that

$$\mathcal{F}_t \subset \widehat{\mathcal{F}}_t := \bigcap_{Q\in\mathcal{P}} \mathcal{F}_t^Q \subset \mathcal{F}_t^P,$$

where $\mathcal{P}$ denotes the family of all probability measures on $(\Omega, \mathcal{F}_t)$ and $\mathcal{F}_t^P$ the completion of the σ-algebra $\mathcal{F}_t$ with respect to a probability measure P. Recall that $\mathcal{F}^P = \sigma(\mathcal{F}, \mathcal{N}^P)$, where

$$\mathcal{N}^P := \{N \subset \Omega : \exists A \in \mathcal{F} \text{ s.t. } N \subset A \text{ and } P(A) = 0\}.$$

Thus, assuming that the initial σ-algebra $\mathcal{F}_0$ is complete with respect to P gives $\mathcal{F}_0 = \widehat{\mathcal{F}}_0$, so, $\{\tau < t\} \in \widehat{\mathcal{F}}_t$, and if we now assume that the filtration is right-continuous, i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$ for every $t \in \mathbb{R}_+$, we get from (1) that τ is a stopping time. A filtration satisfying $\mathcal{F}_0 = \mathcal{F}_0^P$ and $\mathcal{F}_t = \mathcal{F}_{t+}$ for every $t \in \mathbb{R}_+$ is said to satisfy the *usual conditions*.

Solution 1.3

```python
import numpy
from pylab import hist, show
from brownian import brownian


def main():

  # The Wiener process parameter.
  delta = 1
  # Total time.
  T = 1.0
  # Number of steps.
  N = 500
  # Time step size
  dt = T/N
  # Number of realizations to generate.
  m = 5000
  # Create an empty array to store the realizations.
  x = numpy.empty((m,N+1))
  # Initial values of x.
  x[:, 0] = 0

  # Simulate the paths
  brownian(x[:,0], N, dt, delta, out=x[:,1:])

  # Stop the paths for b)
  for i in range(m):
    for j in range(N):
      if x[i,j] >= 1:
        x[i,j:] = x[i,j]
        break

  # Print the mean
  print x.mean(axis=0)[N]

  # Plot the terminal distribution
  hist(x[:,N],bins='auto')
  show()


if __name__ == "__main__":
  main()
```