Mathematical Finance

Solution sheet 2

Solution 2.1

(a) By the tower property of conditional expectation and the assumption $E[Y \mid X] = X$ a.s., we have

$$E[X(X-Y)] = E[E[X(X-Y) \mid X]] = E[X^2 - XE[Y \mid X]] = 0.$$

Remark that here we do need the fact that both X and Y are square-integrable. Similarly, we get

$$E[Y(Y-X)] = 0.$$

Consequently,

$$E[(X-Y)^2] = E[X(X-Y)] - E[Y(X-Y)] = 0,$$

which is sufficient to conclude that $X = Y$ a.s.. Indeed, for any non-negative random variable Z with $E[Z] = 0$, we have that $Z = 0$ a.s.. Choosing $Z := (X-Y)^2$ yields the desired result.

(b) We define $f(x) = \arctan(x)$ for all $x \in \mathbb{R}$. Provided by the fact that the function f is bounded, we have

$$E[f(X)(X-Y)] = E[E[f(X)X - f(X)Y \mid X]] = E[f(X)X] - E[f(X)X] = 0.$$

Similarly,

$$E[f(Y)(Y-X)] = 0.$$

Consequently,

$$E[(f(X) - f(Y))(X-Y)] = E[(f(X)(X-Y)] - E[f(Y)(X-Y)] = 0.$$

Since the function f is increasing, we have that $Z := (f(X) - f(Y))(X-Y) \geqslant 0$ a.s. and we may conclude as above that $X = Y$ a.s..

Solution 2.2 Remark that, for $(X^n)_{n\in\mathbb{N}} \subset \mathbb{D}$, if

$$P(\sup_{s\leqslant t} |X^n_s - X_s| > K) \to 0$$

as $n \to \infty$, for every $t, K > 0$, then

$$E[M \wedge \sup_{s\leqslant t} |X^n_s - X_s|] \to 0$$

as $n \to \infty$, for every $M > 0$, so, the convergence in the metric d is indeed equivalent to the uniform convergence on compacts in probability.

(a) Let $(X^n)_{n\in\mathbb{N}}$ of $\mathbb{D}$ (or $\mathbb{L}$) be a Cauchy sequence in the u.c.p. metric d, i.e., $d(X^m, X^n) \to 0$ as $m, n \to 0$. We may extract a subsequence $Y^k = X^{n_k}$ such that

$$d(Y^m, Y^n) = \sum_{k=1}^{\infty} E\left[2^{-k} \wedge \sup_{t\leqslant k} |Y^m_t - Y^n_t|\right] \leqslant 2^{-n}$$

for all $m \geqslant n$. Then, for non-negative summands, we have

$$E\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} 2^{-k} \wedge \sup_{t\leqslant k}|Y_t^{n+1} - Y_t^n|\right] = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E\left[2^{-k} \wedge \sup_{t\leqslant k}|Y_t^{n+1} - Y_t^n|\right] \leqslant 1$$

and we conclude that the sum inside the expectation on the left is finite almost surely, so, almost surely, the paths of the sequence $(X^n)_{n\in\mathbb{N}}$ are Cauchy convergent in the convergence of uniformly on compacts along a subsequence $(Y^n)_{n\in\mathbb{N}}$ and there is an almost surely unique limit X. The limit is measurable and we have

$$X_t = \lim_{n\to\infty} Y_t^n \tag{1}$$

for every $t \in \mathbb{R}_+$, so, the process X is adapted. Moreover, due to the uniform convergence, we have $X_{t+} = \lim_{n\to\infty} Y_{t+}^n$ and $X_{t-} = \lim_{n\to\infty} Y_{t-}^n$, so, by (1), if $(X^n)_{n\in\mathbb{N}}$ are càdlàg (resp. càglàd), then X is càdlàg (resp. càglàd).

(b) Since J_X and I_{X^t}, $t \in [0,\infty[$, are linear mappings between linear topological spaces, it suffices to establish the continuity at the origin of $\mathbb{S}$. Let X be a good integrator and $H^n \to 0$ uniformly. Then, as $H^n \overset{u.c.p.}{\to} 0$, by the definition of a good integrator, we have $J_X(H^n) \overset{u.c.p.}{\to} 0$, so, in particular, for any $t \in [0,\infty[$, we have $I_{X^t}(H^n) \to 0$ in probability as claimed. For contrary, assume that I_{X^t} is continuous in the uniform norm metric for every $t \in [0,\infty[$ and let $H^n \overset{u.c.p.}{\to} 0$. For every $K, \varepsilon > 0$, we find $\delta > 0$ such that $P\left(\sup_{s\leqslant t}|J_X(H)_s| > K\right) \leqslant \varepsilon/2$ whenever $||H||_\infty \leqslant \delta$. For each $n \in \mathbb{N}$, define $\widetilde{H}^n := H^n \mathbb{1}_{[0,\tau_n]} 1_{\{\tau_n>0\}}$ for $\tau_n := \inf\{t \in \mathbb{R}_+ : |H_t^n| > \delta\}$. We have $\widetilde{H}^k \in \mathbb{S}$ and $||\widetilde{H}^k||_\infty \leqslant \delta$ by left continuity. Moreover, we have $J_X(\widetilde{H}^n)_t = J_X(H^n)_t$ on $\{t \leqslant \tau_n\}$ for every $n \in \mathbb{N}$, so, we get

$$\begin{aligned} P\left(\sup_{s\leqslant t}|J_X(H^n)_s| > K\right) &\leqslant P\left(\sup_{s\leqslant t}|J_X(\widetilde{H}^n)_s| > K\right) + P(\tau_n < t) \\ &\leqslant \varepsilon/2 + P\left(\sup_{s\leqslant t}|H_s^n| > K\right) < \varepsilon \end{aligned}$$

for n sufficiently large, since $\lim_{n\to\infty} P\left(\sup_{s\leqslant t}|H_s^n| > K\right) = 0$. So, J_X is continuous in the u.c.p. topology on $\mathbb{S}$, i.e., X is a good integrator.

Solution 2.3

```
import numpy
from pylab import hist, show
from math import ceil
from brownian import brownian


def main():

  # The Wiener process parameter.
  delta = 1
  # Total time.
  T = 100.0
  # Number of steps.
  N = 500
  # Time step size
  dt = T/N
```

```python
# Number of realizations to generate.
m = 5000
# Create an empty array to store the realizations.
x = numpy.empty((m,N+1))
# Initial values of x.
x[:, 0] = 0
# Simulate the paths
brownian(x[:,0], N, dt, delta, out=x[:,1:])

# Define the put option
def f(x):
  return 1-x if x < 1 else 0

# Compute and print the value of the put
f = numpy.vectorize(f)
p = f(x[:,N]).mean(axis=0)
print 'Value of the put: %f' % p

# Evaluate the strategy
a = numpy.empty((m,N+1))
a[:,0]=-1
b = numpy.empty((m,N+1))
h = True
for i in range(m):
  for j in range(N-1):
    if h == True:
      if x[i,j] >= 2:
        b[i,j] = 1
        h = False
    else:
      if x[i,j] <= 1:
        a[i,j] = -1
        h = True
  if h == True:
    b[i,N]=1

# Compute and print the # of upcrossings
u = numpy.sum(b[:,:N-1])/m
print 'Upcrossings: %f' % u


# Compute and print the value of the portfolio
v = numpy.mean(numpy.sum(numpy.multiply(x,b+a),axis=1))
print 'Value of the portfolio: %f' % v

# Check
print "Doob's upcrossing inequality is: %r" % (u <= p + v)

# Plot the distribution of the portfolio value
hist(numpy.sum(numpy.multiply(x,b+a),axis=1), bins='auto')
show()

```

```
if __name__ == "__main__":
  main()
```