Mathematical Finance

Solution sheet 3

Solution 3.1 Let $\mathcal{C}$ be a class of processes. We write $X \in \mathcal{C}_{loc}$ if there exists a sequence (τ_n) of stopping times with $\tau_n \uparrow \infty$ such that for each n the stopped process X^{τ_n} is in $\mathcal{C}$.

(a) Every good integrator is a locally good integrator; choose a localizing sequence $\tau_n \equiv \infty$. For the converse, let (τ^n) be a localizing sequence for X and choose n such that $P(\tau^n \leqslant t) < \varepsilon/2$. Then, for $(H^k) \subset \mathbb{S}$ such that $||H^k||_\infty \to 0$, for any $K > 0$, we have

$$\begin{aligned} P(|J_X(H^k)_t| > K) &\leqslant P(|J_{X^{\tau^n}}(H^k)_t| > K) + P(\tau^n \leqslant t) \\ &\leqslant P(|J_{X^{\tau^n}}(H^k)_t| > K) + \varepsilon/2 < \varepsilon \end{aligned}$$

for k sufficiently large, since X^{τ^n} is a good integrator.

(b) Since $EY_1 = 1 \neq EY_0 = 0$, the process Y is not a martingale. To show that it is a local martingale, choose $\tau_n = \inf\{t \geqslant 0 : |Y_t| \geqslant n\} \wedge n$, $n \geqslant 1$. Let $s < t < 1$ and $A \in \mathcal{F}_s$. We have

$$E[\mathbb{1}_A(Y_t^{\tau_n} - Y_s^{\tau_n})] = E[\mathbb{1}_A(Y_{(\tau^n \wedge t)\vee s} - Y_s)].$$

By the optional stopping theorem, $(Y_t^{\tau_n})_{t\geqslant 0}$ is a uniformly bounded martingale on $t < 1$. Moreover, Y^{τ_n} is continuous at $t = 1$, and constant on $t \geqslant 1$. Therefore, $(Y_t^{\tau_n})_{t\geqslant 0}$ is martingale, for every $n \geqslant 1$. This shows that Y is a local martingale.

Solution 3.2

(a) The family of càglàd functions $L([0,1])$ equipped with the uniform norm is a Banach space. For $y \in L([0,1])$, let

$$T_n(y) := \sum_{t_k^n, t_{k+1}^n \in \pi^n} y(t_k^n)(x(t_{k+1}^n) - x(t_k^n)).$$

For each n, we find $y \in L([0,1])$ such that $y(t_k^n) = \operatorname{sign}\Big(x(t_{k+1}^n) - x(t_k^n)\Big)$ and $||y||_\infty = 1$. For such y we have

$$T_n(y) = \sum_{t_k^n, t_{k+1}^n \in \pi^n} |x(t_{k+1}^n) - x(t_k^n)|.$$

Therefore

$$||T_n|| \geqslant \sum_{t_k^n, t_{k+1}^n \in \pi^n} |x(t_{k+1}^n) - x(t_k^n)|,$$

for each n, and

$$\sup_n ||T_n|| \geqslant \text{ the total variation of } x.$$

On the other hand, for each $y \in L([0,1])$, $\lim_{n\to\infty} T_n(y)$ exists and therefore $\sup_n |T_n(y)| < \infty$. By Banach-Steinhaus theorem, we have $\sup_n ||T_n|| < \infty$, and the total variation of x is finite.

(b) Recall that a sequence of partitions tending to identity consist collections of stopping times $\Pi^n := \{0 = \tau_0^n < \cdots < \tau_k^n < \infty$ such that $\sup_k |\tau_{k+1}^n - \tau_k^n| \to 0$ and $\sup_k \tau_k^n \to \infty$ as $n \to \infty\}$. For such Π^n and $Z \in \mathbb{L}$, denote

$$Z^{\Pi^n} := \sum_{\tau_k^n, \tau_{k+1}^n \in \Pi^n} Z_{\tau_k^n} \mathbb{1}_{]\tau_k^n, \tau_{k+1}^n]}.$$

Since $\mathbb{S}$ is an u.c.p. dense subset of $\mathbb{L}$ and $Y \in \mathbb{L}$, we find a sequence (Y^m) of bounded simple processes such that $Y^m \overset{u.c.p.}{\to} Y$. We have

$$((Y - Y^{\Pi^n}) \bullet X) = ((Y - Y^m) \bullet X) + ((Y^m - (Y^m)^{\Pi^n}) \bullet X) + (((Y^m)^{\Pi^n} - Y^{\Pi^n}) \bullet X).$$

Since $d((Y^m)^{\Pi^n}, Y^{\Pi^n}) \leqslant d(Y^m, Y)$ for all n, by the continuity of $Y \mapsto (Y \bullet X)$, we have $((Y^m - Y) \bullet X) \overset{u.c.p.}{\to} 0$ and $(((Y^m)^{\Pi^n} - Y^{\Pi^n}) \bullet X) \overset{u.c.p.}{\to} 0$ uniformly in n as $m \to \infty$. For, m and ω fixed, let

$$Y^m_s(\omega) = Y^m_0(\omega) + \sum_{i=1}^{M} Y_{\sigma_i(\omega)} 1_{]\sigma_i(\omega), \sigma_{i+1}(\omega)]}(s).$$

Then

$$(Y^m)^{\Pi^n}_s(\omega) := Y^m_0(\omega) + \sum_{j=1}^{k} Y^m_{\tau^n_j(\omega)} 1_{]\tau^n_j(\omega), \tau^n_{j+1}(\omega)]}(s)$$

and denote $t^n_i(\omega) := \inf\{\tau^n_k(\omega) : \tau^n_k(\omega) > \sigma_i(\omega)\}$ for each $i = 1, \ldots, M$. We have

$$\sup_{s \leqslant t} |((Y^m - (Y^m)^{\Pi^n}) \bullet X)_s(\omega)| \leqslant \sum_{i=1}^{M} |Y_{\sigma_{i+1}(\omega)}(\omega) - Y_{\sigma_i(\omega)}(\omega)| \sup_{\sigma_i(\omega) < s \leqslant t^n_i(\omega)} |X_s(\omega)|.$$

By the right-continuity of $X(\omega)$ and the fact that Y^m is bounded simple, for any $t > 0$, we have

$$\sup_{s \leqslant t} |((Y^m - (Y^m)^{\Pi^n}) \bullet X)_s|(\omega) \to 0 \text{ as } n \to \infty$$

since the mesh $\sup_k |\tau^n_{k+1}(\omega) - \tau^n_k(\omega)|$ and consequently $\sup_i |t^n_i(\omega) - \sigma_i(\omega)|$ tends to zero when $n \to \infty$. In particular, for any m, we have

$$((Y^m - (Y^m)^{\Pi^n}) \bullet X) \overset{u.c.p.}{\to} 0$$

as $n \to \infty$.

Solution 3.3 For a process Z with dynamics $dZ = Z_- dX$, the continuous part of the quadratic variation satisfies $d[Z, Z]^c = Z_- d[X, X]^c$ and the jumps are $\Delta Z = Z_- \Delta X$. By Itô's formula,

$$\begin{aligned} d\log(Z) &= Z_-^{-1} dZ - \frac{1}{2} Z_-^{-2} d[Z, Z]^c + (\Delta \log(Z) - Z_-^{-1} \Delta Z) \\ &= dX - \frac{1}{2} d[X, X]^c + (\log(1 + \Delta X) - \Delta X), \end{aligned} \tag{1}$$

where we did use the identity

$$\Delta \log(Z) = \log(1 + Z_-^{-1} \Delta Z) = \log(1 + \Delta X).$$

Integrating (1) gives

$$\log(Z_t) = X_t - X_0 - \frac{1}{2}[X, X]^c_t + \sum_{s \leqslant t} \left(\log(1 + \Delta X_s) - \Delta X_s\right).$$

Exponenting gives

$$Z_t = \exp\left(X_t - \frac{1}{2}[X, X]_t\right) \Pi_{s \leqslant t}(1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right)$$

with $Z_0 = 1$ as claimed.

Solution 3.4

```python
import numpy
import matplotlib.pyplot as plt

from poisson import poisson


def main():

    # The Poisson process parameter.
    intensity = 1000
    # Total time.
    T = 1.0
    # Number of steps.
    N = 10000
    # Time step size
    dt = T/N
    # Number of realizations to generate.
    m = 5
    # Create an empty array to store the realizations.
    x = numpy.empty((m,N+1))
    # Initial values of x.
    x[:, 0] = 0

    poisson(x[:,0], N, dt, intensity, compensated=True, out=x[:,1:])

    t = numpy.linspace(0.0, N*dt, N+1)
    for k in range(m):
        plt.step(t, x[k])
    plt.show()


if __name__ == "__main__":
    main()
```