Mathematical Finance

Solution sheet 4

Solution 4.1 The covariation of X and A can be expressed as a Lebesgue-Stieltjes integral

$$\begin{aligned}
[X, A]_t &= \lim_n \sum_{k=1}^n (X_{kt/n} - X_{(k-1)t/n})(A_{kt/n} - A_{(k-1)t/n}) \\
&= \lim_n \sum_{k=1}^n \int_0^t \mathbb{1}_{\{(k-1)t/n < s \leqslant kt/n\}} (X_{kt/n} - X_{(k-1)t/n}) dA_s \\
&= \int_0^t \lim_n \sum_{k=1}^n \mathbb{1}_{\{(k-1)t/n < s \leqslant kt/n\}} (X_{kt/n} - X_{(k-1)t/n}) dA_s \\
&= \int_0^t \Delta X_s dA_s,
\end{aligned}$$

where we used the boundedness of X. For each ω, the set $S_t(\omega) = \{s \leqslant t : \Delta X(\omega)_s \neq 0\}$ is at most countable, and the boundedness of X can be used to evaluate the integral

$$\begin{aligned}
\int_0^t \Delta X_s dA_s &= \int_0^t \sum_{s \in S_t} \Delta X_s \mathbb{1}_{\{s=u\}} dA_u \\
&= \sum_{s \in S_t} \Delta X_s \int_0^t \mathbb{1}_{\{s=u\}} dA_u \\
&= \sum_{s \leqslant t} \Delta X_s \Delta A_s.
\end{aligned}$$

Solution 4.2 For $\lambda \in \mathbb{R}$ and

$$M_t := \exp\left(iX_t + \frac{1}{2}|\lambda|^2 t\right),$$

by Itô's formula,

$$\begin{aligned}
M_t &= M_0 + \int_0^t iM_s dX_s + \int_0^t M_s \frac{1}{2}|\lambda|^2 ds + \frac{1}{2}\int_0^t M_s i^2 \lambda^2 d[X, X]_s \\
&= M_0 + \int_0^t iM_s dX_s.
\end{aligned}$$

Since

$$\sup_{s \leqslant t} |M_s| = \exp(\frac{1}{2}|\lambda|^2 t) < \infty,$$

the process M is a martingale, and from

$$E\left[\frac{M_t}{M_s} \mid \mathcal{F}_s\right] = 1,$$

we obtain

$$E\left[e^{i\lambda(X_t - X_s)} \mid \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}|\lambda|^2 (t-s)\right),$$

where the right-hand side is the characteristic function of the normal distribution $\mathcal{N}(0, t-s)$. The process X is a Brownian motion.

Solution 4.3 Denote $\sup_n E[|M_1^{\pi^n}|^2] < \infty$ by C and $\bigcup_n \pi^n$ by D. The family $\left\{\frac{1}{C}\mathbb{1}_{\{C\neq 0\}} M_1^{\pi^n}\right\}$ is contained in the unit ball of $L^2 := L^2(\Omega, \mathcal{F}_1, P)$, so, by Banach-Alaoglu theorem (in conjunction with Eberlein-Šmulian theorem), there exists a subsequence (π^{n_k}) such that $(M_1^{\pi^{n_k}})$ converges weakly to some $\xi \in L^2$. We have $M_t^{\Pi^{n_k}} \to M_t$ for $t \in D$. On the other hand, for every $t \in D$, we have $M_t^{\Pi^{n_k}} = E[M_1^{\Pi^{n_k}} \mid \mathcal{F}_t] \to E[\xi \mid \mathcal{F}_t]$ weakly. Two limits must coincide, so, we have $M_t = E[\xi \mid \mathcal{F}_t]$, $t \in D$. Let $0 \leqslant s < t < 1$ and $(s_n), (t_n) \subset D$ be such that $s < s_n < t < t_n$, $s_n \downarrow s$ and $t_n \downarrow t$. For any $A \in \mathcal{F}_s$, we have

$$E[M_{s_n}\mathbb{1}_A] = E[M_{t_n}\mathbb{1}_A]. \tag{1}$$

Because (M_{s_n}) and (M_{t_n}) are uniformly integrable, letting $n \to \infty$ in both sides of (1) yields

$$E[M_s\mathbb{1}_A] = E[M_t\mathbb{1}_A],$$

i.e., M is a martingale on $[0,1[$ with $M_t = E[\xi \mid \mathcal{F}_t]$ for $t \in [0,1[$.

Solution 4.4

```python
import numpy

from poisson import poisson
from brownian import brownian


#Define a function to compute the quadratic variation of rows of mxN-matrix x
def qv(x,m,N):

    y = numpy.empty(x.shape)

    for i in range(m):
      for j in range(N):
        y[i,j+1]=y[i,j]+(x[i,j+1]-x[i,j])**2
    return y.mean(axis=0)[N]


def main():

    # The Wiener process parameter.
    delta = 1
    # The Poisson process parameter.
    intensity = 1
    # Total time.
    T = 1.0
    # Number of steps.
    N = 1000
    # Time step size
    dt = T/N
    # Number of realizations to generate.
    m = 1000
    # Create empty arrays to store the realizations.
    x = numpy.empty((m,N+1))
    y = numpy.empty((m,N+1))
```

```
    z = numpy.empty((m,N+1))
    # Initial values of x,y,z.
    x[:, 0] = 0
    y[:, 0] = 0
    z[:, 0] = 0

    # Simulate the paths
    brownian(x[:,0], N, dt, delta, out=x[:,1:])
    poisson(y[:,0], N, dt, intensity, compensated=False, out=y[:,1:])
    poisson(z[:,0], N, dt, intensity, compensated=True, out=z[:,1:])

    print qv(x,m,N), qv(y,m,N), qv(z,m,N)

if __name__ == "__main__":
    main()
```