Mathematical Finance

Solution sheet 5

Remark that, for $(X^n) \subset \mathcal{S}$ and $X \in \mathcal{S}$, we have

$$d_E(X^n, X) \to 0$$

if and only if

$$(H^n \bullet (X^n - X)) \overset{u.c.p.}{\to} 0 \text{ for any } (H^n) \subset \mathbb{S}_1.$$

Solution 5.1

(a) Let $X, Y \in \mathcal{S}$. We have $d_E(X+Y, 0) \leqslant d_E(X, 0) + d_E(Y, 0)$ so that addition is jointly continuous and $d_E(cX, 0) \leqslant d_E(X, 0)$ for real $|c| \leqslant 1$; to show that scalar multiplication is jointly continuous we must show that $d(c^n X, 0) \to 0$ for a sequence of real numbers $|c^n| \leqslant 1$ with $c^n \to 0$. This follows from the fact that X is a good integrator as

$$c^n \to 0 \implies c^n H^n \overset{u.c.p.}{\to} 0 \implies ((c^n H^n) \bullet X) \overset{u.c.p.}{\to} 0$$

for any $(H^n) \subset \mathbb{S}_1$.

(b) The metric d_E is stronger than the metric d of the u.c.p. topology. By the completeness of d for $\mathbb{D}$, a Cauchy sequence (X^n) in the metric d_E converges to a càdlàg process X. It suffices to show that the limit is a semimartingale (a good integrator). We have

$$\begin{aligned} P(|(H \bullet X)_t| > K) &\leqslant P(|(H \bullet (X - X^n))_t| > K) + P(|(H \bullet X)_t| > K) \\ &\leqslant \frac{1}{K} d_E(X, X^n) + P(|(H \bullet X^n)_t| > K). \end{aligned}$$

Since each X^n is a good integrator, the image of $\mathbb{S}_1$ under $H \mapsto (H \bullet X^n)_t$ is bounded in probability. Consequently, the image of $\mathbb{S}_1$ under $H \mapsto (H \bullet X)_t$ is bounded in probability, i.e., X is a good integrator.

Solution 5.2 Let $P \sim Q$. Due to the equivalence, for any $n \in \mathbb{N}$, for every $\varepsilon > 0$, we may find $C > 0$ such that

$$E_P[1 \wedge |Y|] \leqslant E_P[|Y|] \leqslant \frac{1}{2^n} \implies E_Q\left[1 \wedge \frac{|Y|}{C}\right] \leqslant \varepsilon. \tag{1}$$

Given a Cauchy sequence (X_n) in the Emery topology of $(\Omega, \mathcal{F}, P)$, we find, for some $2^L > C$, that

$$\sup_{H \in \mathbb{S}_1} E_P[1 \wedge 2^L \sup_{s \leqslant t} |(H \bullet (X^n - X^m)_s|] \leqslant \frac{1}{2^N} \text{ for } m, n \geqslant N + L,$$

so, from (1), we get

$$\sup_{H \in \mathbb{S}_1} E_Q[1 \wedge \sup_{s \leqslant t} |(H \bullet (X^n - X^m)_s|] \leqslant \varepsilon \text{ for } m, n \geqslant N + L,$$

i.e., (X_n) is Cauchy in the Emery topology of $(\Omega, \mathcal{F}, Q)$.

Solution 5.3 Let $(Y^n) \subset \mathbb{L}$ such that $Y^n \overset{u.c.p.}{\to} 0$ and $(H^n) \subset \mathbb{S}_1$. Then $H^n Y^n \overset{u.c.p.}{\to} 0$ and consequently

$$(H^n \bullet (Y^n \bullet X)) = ((H^n Y^n) \bullet X) \overset{u.c.p.}{\to} 0,$$

i.e., $(Y^n \bullet X) \to 0$ in the Emery topology.

Solution 5.4

```
import numpy
from pylab import hist, show
from matplotlib.pyplot import subplot

from brownian import brownian


#Function computes the forward integral of rows of a mxN—matrix w.r.t. another.
def integral(x,y,m,N,out=None):

    if out is None:
        out = numpy.empty(x.shape)

    for i in range(m):
  for j in range(N):
    out[i,j+1]=out[i,j]+x[i,j]*(y[i,j+1]—y[i,j])

    return out


def main():

    # The Wiener process parameter.
    delta = 1
    # Total time.
    T = 1.0
    # Number of steps.
    N = 1000
    # Time step size
    dt = T/N
    # Number of realizations to generate.
    m = 5000
    # Create empty arrays to store the realizations and integrals.
    x = numpy.empty((m,N+1))
    y = numpy.empty((m,N+1))
    z = numpy.empty((m,N+1))
    w = numpy.empty((m,N+1))
    # Initial values of x,y,z,w.
    x[:, 0] = 0
    y[:, 0] = 0
    z[:, 0] = 0
    w[:, 0] = 0

    # Simulate the paths
    brownian(x[:,0], N, dt, delta, out=x[:,1:])
    brownian(y[:,0], N, dt, delta, out=y[:,1:])

    # Compute the integrals
    integral(x,y,m,N,out=z)
```

```
    integral(y,z,m,N,out=w)

    # Plot the terminal distribution
    hist(0.5*(z[:,N]-w[:,N]),normed=True,bins='auto')
    show()

if __name__ == "__main__":
    main()
```