Mathematical Finance

Solution sheet 12

Solution 12.1 For $\Omega = \{\omega_1, \ldots, \omega_N\}$, denoting $X_T(\omega_n) := \xi_n$, $p_n = P(\omega_n)$ and $q_n^m = Q^m(\omega_n)$, the utility maximization problem is

$$\text{maximize } E_P[U(X_T)] = \sum_{n=1}^{N} p_n U(\xi_n)$$

subject to

$$E_{Q^m}[X_T] = \sum_{n=1}^{N} q_n^m \xi_n \leqslant x, \; m = 1, \ldots, M,$$

where $\{Q^m : m = 1, \ldots, M\}$ are the extreme points of $\mathcal{M}^a(s)$. The Lagrangian is

$$\begin{aligned} L(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_M) &= \sum_{n=1}^{N} p_n U(\xi_n) - \sum_{m=1}^{M} \eta_m \left(\sum_{n=1}^{N} q_n^m \xi_n - x \right) \\ &= \sum_{n=1}^{N} p_n \left(U(\xi_n) - \sum_{m=1}^{M} \eta_m \frac{q_n^m}{p_n} \xi_n \right) + \sum_{m=1}^{M} \eta_m x. \end{aligned}$$

Writing $y := \eta_1 + \cdots + \eta_M$, $\mu_M := \frac{\eta_m}{y}$, $\mu := (\mu_1, \ldots, \mu_m)$ and $Q^\mu := \mu_1 Q^1 + \cdots + \mu_M Q^M$, we get

$$L(\xi_1, \ldots, \xi_N, y, Q) = E_P[U(X_T)] - y E_Q[X_T - x] = \sum_{n=1}^{N} p_n \left(U(\xi_n) - y \frac{q_n}{p_n} \xi_n \right) + yx.$$

Write

$$\Phi(\xi_1, \ldots, \xi_n) := \inf_{y>0, Q \in \mathcal{M}^a(S)} L(\xi_1, \ldots, \xi_N, y, Q) \text{ and } \Psi(y, Q) := \inf_{\xi_1, \ldots, \xi_N} L(\xi_1, \ldots, \xi_N, y, Q)$$

so that

$$u(x) = \sup_{\xi_1, \ldots, \xi_N} \Phi(\xi_1, \ldots, \xi_n).$$

On the dual,

$$\Psi(y, Q) = \sum_{n=1}^{N} p_n V(y \frac{q_n}{p_n}) + yx,$$

minimizing subject to Q, for each $y > 0$, we obtain the dual value function

$$v(y) = \inf_{Q \in \mathcal{M}^a(S)} \sum_{n=1}^{N} p_n V\left(y \frac{q_n}{p_n} \right) = \sum_{n=1}^{N} p_n V\left(y \frac{\widehat{q}_n(y)}{p_n} \right), \tag{1}$$

where the optimizer $\widehat{Q} = (\widehat{q}_1, \ldots, \widehat{q}_N)$ exists due to compactness of $\mathcal{M}^a$. Thus, as in the complete case, for

$$\widehat{y} := \operatorname*{argmin}_{y>0} \{v(y) + yx\}$$

and the corresponding $\widehat{Q} := (\widehat{q}_1, \dots, \widehat{q}_N)$ given by (1), we get $\widehat{X}_T := (\widehat{\xi}_1, \dots, \widehat{\xi}_N)$ as

$$\widehat{\xi}_n := [U']^{-1}\left(\widehat{y}\frac{\widehat{q}_n}{p_n}\right)$$

and $\left(\widehat{X}_T, \widehat{y}, \widehat{Q}\right)$ is the unique saddle-point of the Lagrangian and the value functions u and v are conjugate to each other.

Solution 12.2

(a) Consider $x^1, x^2 \in \mathbb{R}$, $\lambda \in [0,1]$, and $\alpha^1, \alpha^2 \in \mathcal{A}$. We write $x^\lambda := \lambda x^1 + (1-\lambda)x^2$. Also X^{t,x^i} is the wealth process starting from x^i at time t and controlled by α^i, where $i \in \{1,2\}$. Set

$$\alpha_s^\lambda := \lambda\alpha_s^1 + (1-\lambda)\alpha_s^2.$$

By the convexity of A, the process α^λ lies in the admissibility class $\mathcal{A}$. Moreover from the linear dynamics of the wealth process, we see that $X^\lambda = \lambda X^{t,x^1} + (1-\lambda)X^{t,x^2}$ is governed by

$$\begin{aligned} dX_s^\lambda &= \alpha_t^\lambda\left((\mu - r)dt + \sigma dW_t\right), \quad s \geqslant t, \\ X_t^\lambda &= x^\lambda. \end{aligned}$$

Therefore, by the concavity of the utility function U, we have

$$U(\lambda X_T^{t,x_1} + (1-\lambda)X_T^{t,x_2}) \geqslant \lambda U(X_T^{t,x_1}) + (1-\lambda)U(X_T^{t,x_2}),$$

which implies that

$$v(\lambda x_1 + (1-\lambda)x_2) \geqslant \lambda U(X_T^{t,x_1}) + (1-\lambda)U(X_T^{t,x_2}).$$

Since the choice of α^1 and α^2 above was arbitrary, we conclude that

$$v(\lambda x_1 + (1-\lambda)x_2) \geqslant \lambda v(x_1) + (1-\lambda)v(x_2).$$

(b) The dynamic programming principle is

$$v(t,x) = \text{ess sup}_{\alpha\in\mathcal{A}}\, E\left[v(\theta, X_\theta^{t,x,\alpha})\middle|\mathcal{F}_t\right],$$

for $\theta \in [t,T]$ and the dynamic programming equation is

$$-v_t(t,x) - \sup_{a\in A}\left\{a(\mu - r)v_x(t,x) + \frac{1}{2}a^2\sigma^2 v_{xx}(t,x)\right\} = 0$$

with the boundary condition $v(T,x) = U(x)$.

Solution 12.3 The pointwise maximization in the HJB leads to the following candidate for the optimal control

$$\widehat{\alpha}(t,x) = -\frac{\mu - r}{\sigma^2}\frac{v_x}{v_{xx}}(t,x) \tag{2}$$

and the HJB becomes

$$-v_t(t,x) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{v_x^2}{v_{xx}}(t,x) = 0.$$

For the exponential utility, we postulate that the solution is of the form

$$v(t,x) = -e^{-\gamma x}\phi(t).$$

Then the HJB reduces to an ODE,

$$\phi'(t) = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\phi(t),$$

whose solution satisfying the terminal condition $\phi(T) = 1$ is

$$\phi(t) = \exp\left(-\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}(T - t)\right),$$

i.e.,

$$v(0, x) = -\exp\left(-\gamma x - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}T\right). \tag{3}$$

From (2), we obtain that the optimal monetary amount invested in the stock is constant

$$\widehat{\alpha}_t = \frac{\mu - r}{\gamma\sigma^2}.$$

By the verification theorem, $\widehat{\alpha}$ is the optimal control and the optimal value is given by (3).

Solution 12.4

```
import numpy
import matplotlib.pyplot as plt

from poisson import poisson
from brownian import brownian


def main():

    # The interest rate.
    r = 0.0
    # The mean value return rate.
    mu = 0.06
    # The Wiener process parameter.
    delta = 0.4
    # Total time.
    T = 1.0
    # Number of steps.
    N = 1000
    # Time step size
    dt = T/N
    # Number of realizations to generate.
    m = 1
    # Create an empty array to store the realizations.
    x = numpy.empty((m,N+1))
    y = numpy.empty((m,N+1))
    z = numpy.empty((m,N+1))
    # Initial values of x,y,z.
    x[:, 0] = 0
    y[:, 0] = 1

    # Simulate the paths
```

```python
    brownian(x[:,0], N, dt, delta, out=x[:,1:])

    # Plot the paths of the wealth process for gamma=10^n, n=0,1,2
    for n in range(3):
        y = (mu-r)/(10.**n*delta**2.)*(x+numpy.cumsum(dt*numpy.ones((m,N+1))))
        t = numpy.linspace(0.0, N*dt, N+1)
    for k in range(m):
          plt.step(t, y[k])
    plt.show()



if __name__ == "__main__":
    main()
```