Mathematical Finance

Solution sheet 14

Solution 14.1 Denote by $Z = (Z_t)_{t\in[0,T]}$ the density process process of Q with respect to P.

(a) The second claim follows directly from the first claim together with the fact that $yZ_T = y\frac{dQ}{dP} \in \mathcal{D}(y)$ since $Z \in \mathcal{Z}(1)$ and the fact that the function V is decreasing. So it remains to show the first claim. Seeking a contradiction, suppose there exists $Y_T \in \mathcal{D}(y)$ such that $A := \{Y_T > yZ_T\}$ has $P[A] > 0$. Set $a = Q[A] > 0$ and define the Q-martingale $M = (M_t)_{t\in[0,T]}$ by $M_t := E_Q[1_A \mid \mathcal{F}_t]$. Then M is non-negative and $M_0 = a$ by the fact that $\mathcal{F}_0$ is P-trivial. By the predictable representation property of S under Q, there exists $H \in L(S)$ such that $M = a + H \bullet S$. Thus, $M \in \mathcal{V}(a)$. Now, on the one hand, by the definition of $\mathcal{D}(y)$, there exists a supermartingale $\widetilde{Z} \in \mathcal{Z}(y)$ with $Y_T \leqslant \widetilde{Z}_T$. Therefore,

$$E[M_T Y_T] \leqslant E[M_T \widetilde{Z}_T] \leqslant E[M_0 \widetilde{Z}_0] = ay. \tag{1}$$

On the other hand,

$$E[Z_T M_T] = E_Q[M_T] = M_0 = a. \tag{2}$$

Thus, we arrive at the contradiction

$$0 \geqslant E[M_T(Y_T - yZ_T)] = E[1_{\{Y_T > yZ_T\}}(Y_T - yZ_T)] > 0. \tag{3}$$

(b) Note that $0 \leqslant y_0 < \infty$ and $v(y) < \infty$ on (y_0, ∞). Moreover, recall that the function V is strictly decreasing, strictly convex and in C^1 on $(0, \infty)$.

First, define the function $g : (y_0, \infty) \to [-\infty, 0]$ by

$$g(s) = E[Z_T V'(sZ_T)]. \tag{4}$$

This is well defined as $Z_T > 0$ P-a.s. and $V' < 0$. Moreover, it is increasing as V' is increasing. Thus if $g(s_0) > -\infty$ for some $s_0 > y_0$, it follows by dominated convergence that it is continuous on (s_0, ∞).

Next, for $y_1, y_2 \in (y_0, \infty)$, $y_1 < y_2$, the fundamental theorem of calculus gives

$$V(y_2 Z_T) - V(y_1 Z_T) = \int_{y_1}^{y_2} Z_T V'(sZ_T)\, ds. \tag{5}$$

Now, the left-hand side of (5) is integrable by assumption. Thus, the right-hand side is so, too, and since $V' < 0$, the integrand on the right-hand side is strictly negative, and Fubini's theorem gives

$$v(y_2) - v(y_1) = \int_{y_1}^{y_2} g(s)\, ds. \tag{6}$$

In particular, the function g is finite a.e. on (y_0, ∞), and thus continuous and finite on (y_0, ∞). Now the claim follows from the fundamental theorem of calculus.

(c) First, recall that $X_T \in \mathcal{C}(x)$ if and only if

$$\sup_{Y_T \in \mathcal{D}(1)} E[X_T h] \leqslant x. \tag{7}$$

By part **a)**, this is equivalent to

$$E[X_T Z_T] \leqslant x. \tag{8}$$

Now, by part **b)** and the choice of $\widehat{y}(x)$,

$$E[\widehat{X}_T Z_T] = E[-V'(\widehat{y}(x)Z_T)Z_T] = -v'(\widehat{y}(x)) = x, \tag{9}$$

and so $\widehat{X}_T \in \mathcal{C}(x)$.

Next, fix $X_T \in \mathcal{C}(x)$. We may assume without loss of generality that $E[U(X_T)] > -\infty$. By the fact that $\widehat{X}_T > 0$ P-a.s. and U is in C^1 and strictly concave on $(0,\infty)$,

$$U(X_T) - U(\widehat{X}_T) \leqslant U'(\widehat{X}_T)(X_T - \widehat{X}_T), \tag{10}$$

where the equality is strict on $\{X_T \neq \widehat{X}_T\}$. Taking expectations and using the fact that $U'(-V') = \mathrm{id}$ and (8) and (9) gives

$$E[U(X_T) - U(\widehat{X}_T)] \leqslant E[U'(\widehat{X}_T)(X_T - \widehat{X}_T)] = \widehat{y}(x)E[Z_T(X_T - \widehat{X}_T)] \leqslant 0. \tag{11}$$

If $X_T = \widehat{X}_T$ P-a.s., then both inequalities are trivially equalities, and if $P[X_T \neq \widehat{X}_T] > 0$, then the first inequality is strict.

Solution 14.2 The situation on which the dual optimizer fails to be of the form

$$\widehat{Y}_T := \frac{dQ}{dP}, \tag{12}$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym density for some EσMM Q is given by Exercise 7.4. Consider for instance the logarithmic utility $U(x) = \log(x)$ for which the dual optimization problem is to minimize

$$E[V(Y_T)] = E\left[-\log(Y_T) - 1\right] = -E\left[\log(Y_T)\right] - 1,$$

or equivalently, maximize

$$E\left[\log(Y_T)\right]$$

over $Y_T \in \mathcal{D}(1)$. Take $S := Z^{-1}$, where $Z := X^{\sigma\wedge\tau}$, given by Exercise 7.4, fails to be uniformly integrable martingale and deploy the usual time change $t/(T-t)$ to obtain a finite time-horizon T. Clearly, $Z_T \in \mathcal{D}(1)$, but since $E[Z_T] < 1$ it does not define a probability measure, i.e., it fails be of the form (12). For any $Y_T \in \mathcal{D}(1)$, the process is a supermartingale starting from $Y_0 S_0 = 1$. Hence, by Jensen's inequality, we have

$$E[\log(Y_T)] = E\left[\log\left(\frac{Y_T}{Z_T}\right)\right] + E[\log(Z_T)] \leqslant \log\big(E[Y_T S_T]\big) + E[\log(Z_T)] \leqslant E[\log(Z_T)],$$

i.e.,

$$v(1) = -E[\log(Z_T)] - 1.$$

Solution 14.3

(a) Since $E[f/\hat{f}] \leqslant 1$ holds for all $f \in \mathcal{C}$, by the Markov inequality we have for any $f \in \mathcal{C}$ and any $M > 0$

$$P[\frac{f}{\hat{f}} \geqslant M] \leqslant \frac{1}{M}E[\frac{f}{\hat{f}}] \leqslant \frac{1}{M},$$

which indeed implies that the set $\{f/\hat{f} | f \in \mathcal{C}\}$ is bounded in probability.
Since $\lim_{M\to\infty} P[\hat{f} \geqslant M] = 0$, from the fact that $P[f \geqslant M] = P[\frac{f}{\hat{f}}\hat{f} \geqslant M] \leqslant P[\frac{f}{\hat{f}} \geqslant \sqrt{M}] + P[\hat{f} \geqslant \sqrt{M}]$ it follows that $\mathcal{C}$ is bounded in probability as well.

(b) The convexity and boundedness in probability of $\mathcal{C}'$ follow immediately from the definition of $\mathcal{C}'$ and the corresponding properties of $\mathcal{C}$. So it remains to show that $\mathcal{C}'$ is closed in probability. Let (f_n) be a sequence in $\mathcal{C}'$ convergent to an $f \in L^0_+$ in probability. By passing to a subsequence we can and will assume that (f_n) converges to f almost surely. By definition for each n there is an h_n in $\mathcal{C}$ such that $f_n \leqslant h_n$. Since $\mathcal{C}'$ is convex and bounded in probability, $\mathrm{conv}(h_n | n \geqslant 1) \subset \mathcal{C}'$ is also bounded in probability and we can apply the Komlos lemma for (h_n) to find a sequence $(\widetilde{h}_n)$ such that $\widetilde{h}_n$ is a finite convex combination of h_n, $h_{n+1}, \dots$ and $\widetilde{h}_n$ converges to an h in probability. Again, we will assume that this convergence also holds almost surely. By the closedness of $\mathcal{C}$, we have $h \in \mathcal{C}$. Furthermore, if $\widetilde{h}_n = a_1^n h_n + \dots + a_{N_n}^n h_{N_n}$ for convex weights $a_1^n, \dots, a_{N_n}^n \geqslant 0$, $a_1^n + \dots + a_{N_n}^n = 1$, then as $h_m \geqslant f_m$ for all $m \geqslant 1$ we indeed have $\widetilde{h}_n \geqslant \widetilde{f}_n$ with $\widetilde{f}_n := a_1^n f_n + \dots + a_{N_n}^n f_{N_n}$. On the other hand, it is clear that with $f_n \to f$ almost surely it holds that $\widetilde{f}_n \to f$ almost surely. Combining all arguments above we can conclude that $h = \lim_{n\to\infty} \widetilde{h}_n \geqslant \lim_{n\to\infty} \widetilde{f}_n = f$, which implies that $f \in \mathcal{C}'$.

(c) Now we assume that $\mathcal{C} = \mathcal{C}'$ and $1 \in \mathcal{C}$. Then it is clear that for each $n \geqslant 1$, $1 \in \mathcal{C}^n$ and therefore $\mathcal{C}^n$ is nonempty. Moreover, using the same argument as in **b)** we can easily check that $\mathcal{C}^n$ is closed in probability and convex. Also, since $1 \in \mathcal{C}$, we have $\sup_{f\in\mathcal{C}^n} E[\log f] \geqslant E[\log 1] = 0$. On the other hand, since each $f \in \mathcal{C}^n$ satisfies $f \leqslant n$, we have $\sup_{f\in\mathcal{C}^n} E[\log f] \leqslant \log n$. Now let (f_m) be a sequence in $\mathcal{C}^n$ such that $E[\log f_m] \uparrow \sup_{f\in\mathcal{C}^n} E[\log f]$ as m tends to ∞. Using Komlos lemma for (f_m) we obtain a sequence $(\widetilde{f}_m)$ and an $f^n \in L^0_+$ such that $\widetilde{f}_m$ is a finite convex combination of f_m, $f_{m+1}, \dots$ and $\widetilde{f}_m$ converges to f^n in probability. Since $x \mapsto \log x$ is a concave function, we have $E[\log \widetilde{f}_m] \geqslant E[\log f_m]$ for each m (note that we assume $E[\log f_m]$ is increasing in m) for all m. Moreover, since $\mathcal{C}^n$ is convex and in particular each $\widetilde{f}_m$ is in $\mathcal{C}^n$, it holds that $\widetilde{f}_m \in \mathcal{C}^n$. As a consequence of the closedness of $\mathcal{C}^n$, we have $f^n \in \mathcal{C}^n$ and by the inverse Fatou's lemma (as $\log f \leqslant \log n$ for all $f \in \mathcal{C}^n$) we get

$$E[\log f^n] \geqslant \limsup_{m\to\infty} E[\log \widetilde{f}_m] \geqslant \limsup_{m\to\infty} E[\log f_m] \geqslant \sup_{f\in\mathcal{C}^n} E[\log f].$$

This gives $E[\log f^n] = \sup_{f\in\mathcal{C}^n} E[\log f]$. Finally note that as $E[\log f^n] \geqslant E[\log 1] = 0$ we must have $f^n \in L^0_{++}$.

(d) Now fix an n and we still denote by f^n a maximizer for $\sup_{f\in\mathcal{C}^n} E[\log f]$ as above. For any $\epsilon \in (0, \frac{1}{2}]$ and $f \in \mathcal{C}^n$, since $(1-\epsilon) f^n + \epsilon f$ belongs to $\mathcal{C}^n$ due to its convexity, the maximality of f^n implies that $E[\Delta_\epsilon(f|f^n)] \leqslant 0$ for $\Delta_\epsilon(f|f^n) := \frac{\log\big((1-\epsilon)f^n+\epsilon f\big) - \log f^n}{\epsilon}$. Furthermore, on $\{f > f^n\}$ we indeed have $\Delta_\epsilon(f|f^n) > 0$ while on $\{f \leqslant f^n\}$ we can use the inequality $\log y - \log x \leqslant \frac{y-x}{x}$ for all $0 < x < y$ to show that (set $y = f^n$, $x = (1-\epsilon)f^n + \epsilon f$)

$$\Delta_\epsilon(f|f^n) \geqslant -\frac{f^n - f}{f^n - \epsilon(f^n - f)} \geqslant -\frac{f^n - f}{f^n - \frac{1}{2}(f^n - f)} = -2\frac{f^n - f}{f^n + f} \geqslant -2,$$

where in the second inequality we use the assumption that $\epsilon \leqslant \frac{1}{2}$. Now, by Fatou's lemma, we can derive that

$$E\big[\liminf_{\epsilon\to 0} \Delta_\epsilon(f|f^n)\big] = E\big[\frac{f - f^n}{f^n}\big] \leqslant \liminf_{\epsilon\to 0} E\big[\Delta_\epsilon(f|f^n)\big] \leqslant 0,$$

where in the first equality above we used the fact that $\liminf_{\epsilon\to 0} \big(\log\big((1-\epsilon)x+\epsilon y\big) - \log x\big)/\epsilon = (y-x)/x$. Now we only need to note that $E\big[\frac{f-f^n}{f^n}\big] \leqslant 0$ is equivalent to $E\big[\frac{f}{f^n}\big] \leqslant 1$ and as $h^n \in \mathcal{C}^n \cap L^0_{++}$ (see **c)**) the random variable $1/f^n$ is well-defined.

(e) Now for each n we obtain an $f^n \in \mathcal{C}^n \cap L^0_{++} \subset \mathcal{C}$ (keep in mind we still assume that $\mathcal{C} = \mathcal{C}'$) such that $E[f/f^n] \leqslant 1$ for all $f \in \mathcal{C}^n$. Now, we apply the Komlos lemma for the sequence

(f^n) to get a sequence $(\widetilde{f}^n)$ and an $\hat{f} \in L^0_+$ such that $\widetilde{f}^n$ is a finite convex combination of f^n, $f^{n+1}, \ldots$ and $\widetilde{f}^n$ converges to $\hat{f}$ in probability. The convexity of $\mathcal{C}$ ensures that each $\widetilde{f}^n$ is in $\mathcal{C}$ and therefore the closedness in probability of $\mathcal{C}$ ensures that $\hat{f} \in \mathcal{C}$. Now, for any $f \in \mathcal{C}^k$, for any $n \geqslant k$, since $\mathcal{C}^k \subset \mathcal{C}^n$, we have $E[f/f^n] \leqslant 1$ and, furthermore, as $x \mapsto \frac{1}{x}$ is convex, Jensen's inequality implies that $E[f/\widetilde{f}^n] \leqslant a_1^n E[f/f^n] + \ldots + a^n_{N_n} E[f/f^{N_n}] \leqslant a^n_1 + \ldots + a^n_{N_n} = 1$ for $\widetilde{f}^n := a^n_1 f_n + \ldots + a^n_{N_n} f_{N_n}$ with convex weights $a^n_1, \ldots a^n_{N_n}$. This implies that for all $f \in \bigcup_{k \geqslant 1} \mathcal{C}^k$, by Fatou's lemma we have $E[f/\hat{f}] \leqslant \liminf_{n\to\infty} E[f/\widetilde{f}^n] \leqslant 1$. In particular, inserting $1 \in \bigcup_{k\geqslant 1} \mathcal{C}^k$ we can get $E[1/\hat{f}] \leqslant 1$, which implies that $\hat{f} \in L^0_{++} \cap \mathcal{C}$. Further, Since for any $f \in \mathcal{C}$ we have $f \wedge k \in \mathcal{C}^k$, the monotone convergence theorem provides that

$$E\big[f/\hat{f}\big] = \lim_{k\to\infty} E\big[(f \wedge k)/\hat{f}\big] \leqslant 1.$$

Finally, since $\hat{f} \in \mathcal{C}$ and $\mathcal{C}$ is bounded in probability, we must have $\hat{f} < +\infty$ almost surely. Hence $1/\hat{f}$ is also in L^0_{++}.

(f) For a general $\mathcal{C}$ which is convex, closed and bounded in probability and $\mathcal{C} \cap L^0_{++} \neq \varnothing$, we first pick any $g \in \mathcal{C} \cap L^0_{++}$ and define $\mathcal{C}^g := \big\{f/g | f \in \mathcal{C}\big\}$. Obviously this set $\mathcal{C}^g$ contains the constant 1, and is convex, closed and bounded in probability as well. Now we apply our previous arguments for the solid set $\mathcal{C}^{g,\prime} := \big\{f | f \leqslant h \text{ for some } h \in \mathcal{C}^g\big\}$ and get an element $\hat{h}$ in $\mathcal{C}^{g,\prime}$ such that $E[f/\hat{h}] \leqslant 1$ for all $f \in \mathcal{C}^{g,\prime}$. Clearly this $\hat{h}$ has to be an element in $\mathcal{C}^g$, otherwise we would find a $h \in \mathcal{C}^g$ with $h \geqslant \hat{h}$ and $P[h > \hat{h}] > 0$, which would yield that $E[f/\hat{h}] > 1$. Hence, $\hat{h}$ has the form $\hat{h} = \hat{f}/g$ for some $\hat{f} \in \mathcal{C}$. Finally, since for any $f \in \mathcal{C}$, $E\big[f/\hat{f}\big] = E\big[(\frac{f}{g})/(\frac{\hat{f}}{g})\big] = E\big[(\frac{f}{g})/\hat{h}\big] \leqslant 1$, we see that $\hat{f} \in \mathcal{C}$ satisfies all the requirements we want.

Solution 14.4

(a) The decomposability simply means that the market modeled by $\mathcal{X}$ has the switching property. More precisely, the investors on this market are allowed to switch their portfolios at any time $t \in [0,T]$ from X to X' if the event A, which is observable up to time t, happens, otherwise they will keep their positions.

(b) Let S be a semimartingale. Clearly then any process in $1 + G(\Theta^1_{adm}(S))$ is adapted, RCLL, nonnegative and starts at 1. As 0 is in $\mathcal{X}^1 = \Theta^1_{adm}(S)$, we must have $1 \in \mathcal{X}^1$ which means that $\mathcal{X}^1$ contains a strictly positive process. The convexity follows from the convexity of $G(\Theta^1_{adm}(S))$. To show the decomposability, pick $X = 1 + \vartheta^1 \bullet S$ and $X' := 1 + \vartheta' \bullet S$ from $\mathcal{X}^1$. For any $t \in [0,T]$ and $A \in \mathcal{F}_t$, we define $\widetilde{\vartheta} := 1_{[0,t]}\vartheta + 1_{(t,T]}(1_A \frac{X_t}{X'_t}\vartheta' + 1_{A^c}\vartheta)$. It is easy to see that $\widetilde{\vartheta}$ is predictable and S-integrable. Furthermore, we have

$$1 + (\widetilde{\vartheta} \bullet S)_s = 1_{A^c} X_s + 1_A \frac{X'_{t\vee s}}{X'_t} X_{t\wedge s}$$

for all $s \in [0,T]$, which implies that $\widetilde{\vartheta} \in \Theta^1_{adm}$ and $(1_{A^c}X_s + 1_A \frac{X'_{t\vee s}}{X'_t} X_{t\wedge s})_{s\in[0,T]}$ is in $\mathcal{X}^1$.

(c) Since $\mathcal{X}$ is a wealth process set, it is clear that $\mathcal{X}_T$ is convex and $\mathcal{X}_T \cap L^0_{++} \neq \varnothing$. So, if $\mathcal{X}_T$ is closed in probability and satisfies NUPBR, using the result from the previous exercise we can find an $\widehat{X}_T \in \mathcal{X}_T$ which corresponds to the final value of a wealth process $\widehat{X}$ such that $E[X_T/\widehat{X}_T] \leqslant 1$ holds for all $X \in \mathcal{X}$. Now for any $X \in \mathcal{X}$, for any $t \in [0,T]$, the decomposability ensures that the process

$$Y_s := \frac{\widehat{X}_{t\vee s}}{\widehat{X}_t} X_{t\wedge s}, s \in [0,T]$$

belongs to $\mathcal{X}$ (choose $A = \Omega$). So from $E[Y_T/\widehat{X}_T] \leqslant 1$ it follows that $E[X_t/\widehat{X}_t] \leqslant 1$. Hence, for $t \leqslant r$ in $[0,T]$, $A \in \mathcal{F}_t$, for any strictly positive process $X' \in \mathcal{X}$, since

$$Y_s = 1_{A^c}\widehat{X}_s + 1_A \frac{X'_{t\vee s}}{X'_t}\widehat{X}_{t\wedge s}, s \in [0,T]$$

is an element in $\mathcal{X}$, and $Y_r = 1_{A^c} + 1_A \frac{X'_r}{X'_t}\widehat{X}_t$ the inequality $E[Y_r/\widehat{X}_r] \leqslant 1$ translates into

$$E\big[1_A(X'_r/\widehat{X}_r)/(X'_t/\widehat{X}_t)\big] \leqslant P[A],$$

which in turn implies that $E\big[(X'_r/\widehat{X}_r)/(X'_t/\widehat{X}_t)|\mathcal{F}_t\big] \leqslant 1$ and consequently $X'/\widehat{X}$ is a supermartingale for all strictly positive X' and we can extend this result to all $X \in \mathcal{X}$. Finally note that this $\widehat{X}$ is strictly positive so that $\frac{1}{\widehat{X}}$ is $\mathbb{R}$-valued, and since $\widehat{X}_t < +\infty$, $\frac{1}{\widehat{X}}$ is strictly positive.