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Exercise Series 10

Q1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(Z_n)_{n\in\mathbb{N}}$ a sequence of random variables.

(a) Prove that if $Z_n \xrightarrow{\mathbb{P}} c \in \mathbb{R}$, then for all bounded and continuous functions f

$$\mathbb{E}\left(f(Z_n)\right) \to f(c).$$

(b) Show that if $Z_n \to c \in \mathbb{R}$ in distribution, then $Z_n \xrightarrow{\mathbb{P}} c$.

Q2. Take the following probability space $(\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$, where $\lambda|_{[0,1]}$ is the Lebesgue measure over $[0,1]$. Let $X_n(\omega) = \mathbf{1}_{A_n}(\omega)$ a sequence of random variables with $A_n \in \mathcal{B}([0,1])$.

(a) Under which condition for $(A_n)_{n\in\mathbb{N}}$ we have that $X_n \xrightarrow{\mathbb{P}} 0$.

(b) Write the event $\{\omega : X_n(\omega) \to 0\}$ with help of the sets $(A_n)_{n\in\mathbb{N}}$.

(c) Find a sequence $(A_n)_{n\in\mathbb{N}}$ of events so that $X_n \xrightarrow{\mathbb{P}} 0$ but $\{\omega : X_n(\omega) \to 0\} = \emptyset$.

Q3. Let $(X_i)_{i\geq 1}$ be a sequence of random variables with

$$\begin{aligned}
&\mathbb{E}\left(X_i\right) = \mu \quad \forall i,\\
&Var(X_i) = \sigma^2 < \infty \quad \forall i,\\
&Cov(X_i, X_j) = R(|i-j|) \quad \forall i, j.
\end{aligned}$$

Define $S_n := \sum_{i=1}^n X_i$.

(a) Prove that if $\lim_{k\to\infty} R(k) = 0$ then $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ in probability.

(b) Prove that if $\sum_{k\in\mathbb{N}} |R(k)| < \infty$ then $\lim_{n\to\infty} nVar(\frac{S_n}{n})$ exists.

Q4. **(a)** Let μ_n and ν_n two sequence of probability measure on $\mathbb{R}$. and $\epsilon_n \in (0,1)$ with $\epsilon_n \to 0$. Prove that if $\mu_n \to \mu$ in distribution, then $(1-\epsilon_n)\mu_n + \epsilon_n\nu_n \to \mu$ in distribution.

(b) Construct with the help of a) a sequence μ_n so that $\mu_n \to \mu$ in distribution but $\lim_{n\to\infty} \int |x| d\mu_n(x) \neq \int |x| d\mu(x)$.

(c) Prove that if $\mu_n \to \mu$ in distribution and $\sup_n \int x^2 d\mu_n(x) = K < \infty$ then

$$\int |x| d\mu_n(x) \to \int |x| d\mu(x).$$

Hint: For all M prove that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} d\mu(x).$$

and that

$$0 \leq \int |x| d\mu_n(x) - \int \min\{|x|, M\} d\mu_n(x) \leq K/M.$$