Exercise Sheet 1

Norm, Trace, Discriminant and Rings of Integers

1. Show that $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$.

2. Let k be a field of characteristic $\neq 2$. Let $A := k[x]$ for some transcendental x and let $K := k(x)$ denote its fraction field. Let $L := K[y]/(y^2 - f)$ for some separable $f \in A$ with $\deg f > 0$, and let B be the integral closure of A in L. Show that
$$B = A \oplus A \cdot y.$$

3. Determine the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$ and its discriminant.

4. This is an example by Dedekind of a cubic number field K whose ring of integers is not generated by one element over $\mathbb{Z}$.
   (a) Show that the polynomial $f := X^3 + X^2 - 2X + 8$ is irreducible over $\mathbb{Q}$ and thus defines a cubic number field $K := \mathbb{Q}(\theta)$ with $f(\theta) = 0$.
   (b) Show that the ring of integers of K is $\mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z} \cdot \theta \oplus \mathbb{Z} \cdot \beta$ for $\beta := \frac{\theta + \theta^2}{2}$.
   (c) Show that the kernel of the surjection $\mathbb{Z}[X, Y] \twoheadrightarrow \mathcal{O}_K$ defined by $g(X, Y) \mapsto g(\theta, \beta)$ is the ideal
   $$(X^2 - 2Y + X,\ XY - X + 4,\ Y^2 - Y + 2X + 2).$$
   (d) Deduce that $\mathcal{O}_K / 2\mathcal{O}_K \cong (\mathbb{F}_2)^3$.
   (e) Show that $(\mathbb{F}_2)^3$ is not generated by one element over $\mathbb{F}_2$.
   (f) Deduce that there exists no $\xi \in \mathcal{O}_K$ such that $\mathcal{O}_K = \mathbb{Z}[\xi]$.

5. Two field extensions L/K and L'/K are called *linearly disjoint over* K if $L \otimes_K L'$ is a field. Let $L := \mathbb{Q}(\sqrt[3]{3})$ and let $L' := \mathbb{Q}(\zeta \sqrt[3]{3})$, where ζ is a primitive 3rd root of unity. Show that $L \cap L' = \mathbb{Q}$, but L and L' are not linearly disjoint over $\mathbb{Q}$.

6. Let L/K be an inseparable finite field extension. Then $\mathrm{Tr}_{L/K}$ is identically zero.