Exercise Sheet 2

Dedekind Rings and Lattices

1. Consider the number field $K := \mathbb{Q}(\sqrt{-5})$ and its ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$.

   (a) Show that $(3) = \mathfrak{p}\mathfrak{p}'$ with prime ideals $\mathfrak{p} := (3, 1+\sqrt{-5})$ and $\mathfrak{p}' := (3, 1-\sqrt{-5})$.

   (b) Determine the structure of the ring $\mathcal{O}_K/(3)$.

   (c) Determine the inverse of $\mathfrak{p}$ as a fractional ideal.

   (d) Which powers of the ideal $\mathfrak{p}$ are principal?

   (e) Compute the factorization of (2) into prime ideals.

   (f) Compute the factorization of (5) into prime ideals.

   (g) Compute the factorization of (7) into prime ideals.

2. Let A be a Dedekind domain.

   (a) Show that for any non-zero ideal $\mathfrak{a} \subseteq A$, any ideal of $A/\mathfrak{a}$ is principal.

   (b) Show that every ideal of A is generated by two elements.

3. Show that a subgroup Γ of a finite-dimensional $\mathbb{R}$-vector space V is a complete lattice if and only if Γ is discrete and V/Γ is compact.

4. *(Minkowski's theorem on linear forms)* Let

   $$L_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, n,$$

   be real linear forms such that $\det(a_{ij}) \neq 0$, and let $c_1, \ldots, c_n$ be positive real numbers such that $c_1 \cdots c_n > |\det(a_{ij})|$. Show that there exist integers $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, such that for all $i \in \{1, \ldots, n\}$

   $$|L_i(m_1, \ldots, m_n)| < c_i.$$

   Hint: Use Minkowski's lattice point theorem.

*5. Consider a line $\ell := \mathbb{R} \cdot (1, \alpha)$ in the plane $\mathbb{R}^2$ with an irrational slope $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Show that for any $\varepsilon > 0$, there are infinitely many lattice points $P \in \mathbb{Z}^2$ of distance $d(P, \ell) < \varepsilon$.