Exercise Sheet 5

Units, Decomposition Of Prime Ideals

1. (a) Determine the ring of integers of $K := \mathbb{Q}(\sqrt{5}, i)$.
   (b) Determine $\mathcal{O}_F^\times$ for the subfield $F := \mathbb{Q}(\sqrt{5})$.
   (c) Find a fundamental unit of $\mathcal{O}_K^\times$.
   (d) Show that $|\mu(K)| = 4$ and write down $\mathcal{O}_K^\times$.

2. (a) Let K be a cubic number field with exactly one real embedding. We identify K with its image. Show that for any unit $u \in \mathcal{O}_K^\times$ with $u > 1$ we have
$$|\operatorname{disc}(\mathcal{O}_K)| \leqslant 3\left(u^2 + \frac{2}{u}\right)\left(u^4 + \frac{2}{u^2}\right).$$
   Hint: Use Hadamard's inequality: For any complex $n \times n$-matrix M with columns $v_1, \ldots, v_n$, we have $|\det(M)| \leqslant \prod_{i=1}^n \|v_i\|$.
   (b) Show that a fundamental unit of $\mathcal{O}_K^\times$ for the number field $K := \mathbb{Q}(\sqrt[3]{2})$ is $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

3. Using continued fractions:
   (a) Compute a fundamental unit of $\mathcal{O}_K^\times$ for $K := \mathbb{Q}(\sqrt{318})$.
   (b) Find the smallest positive integer solution of the equation $x^2 - 61y^2 = 1$.

4. Let K be a number field and let S be a finite set of prime ideals of $\mathcal{O}_K$. We define the ring of *S-integers in K* to be
$$\mathcal{O}_{K,S} \;:=\; \bigcap_{\mathfrak{p}\notin S} \mathcal{O}_{K,\mathfrak{p}} \;=\; \big\{\alpha \in K \;\big|\; \forall \mathfrak{p} \notin S : \operatorname{ord}_{\mathfrak{p}}(\alpha) \geqslant 0\big\}.$$
   The group $\mathcal{O}_{K,S}^\times$ is called the group of *S-units in K*.
   (a) Show that the torsion subgroup of $\mathcal{O}_{K,S}^\times$ is $\mu(K)$.
   (b) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the distinct elements of S. Show that the homomorphism
$$\varphi\colon \mathcal{O}_{K,S}^\times \to \mathbb{Z}^t,\ \alpha \mapsto (\operatorname{ord}_{\mathfrak{p}_i}(\alpha))_i$$
   has kernel $\mathcal{O}_K^\times$ and that its image has rank t.
   (c) Deduce that $\mathcal{O}_{K,S}^\times \cong \mu(K) \times \mathbb{Z}^{r+s+|S|-1}$.

5. In the number field $K := \mathbb{Q}(\sqrt[3]{2})$, what are the possible decompositions of $p\mathcal{O}_K$ for rational primes p?