Exercise Sheet 6

Decomposition Of Prime Ideals, Different

1. Let $R := \mathbb{F}_p(t)[x]$ for a rational prime p and algebraically independent t and x. Let A be the localization of R at the prime ideal Rx, and let $\mathfrak{p} := Ax$ denote its maximal ideal. Let K be the quotient field of A.

   (a) Show that the polynomial $f(Y) := Y^p - x^{p-1}Y - t \in A[Y]$ is separable and irreducible over K.

   (b) Consider a field extension $L = K(y)$ with $f(y) = 0$. Show that L/K is galois with Galois group isomorphic to $\mathbb{F}_p$, acting by $y \mapsto y + \alpha x$ for all $\alpha \in \mathbb{F}_p$.

   (c) Show that $B := A[y]$ is the integral closure of A in L and that $\mathfrak{P} := B\mathfrak{p}$ is the unique prime ideal of B above $\mathfrak{p}$.

   (d) Show that the extension of residue fields $k(\mathfrak{P})/k(\mathfrak{p})$ is inseparable.

   *(e) Repeat the constructions of $R, A, \mathfrak{p}, K, L$ after replacing the field $\mathbb{F}_p(t)$ by its inseparable extension $\mathbb{F}_p(s)$ with $s^p = t$, so that $f(Y) = Y^p - x^{p-1}Y - s^p$. Show that (a) and (b) are still true for the resulting items by $R', A', \mathfrak{p}', K', L'$. But in (c) define B' instead as the integral closure of A' in L' and prove that $B'\mathfrak{p}' = \mathfrak{Q}'^p$ for a prime ideal $\mathfrak{Q}'$.

   Note: The correct definition of an unramified prime $\mathfrak{P}/\mathfrak{p}$ requires not only that $e_{\mathfrak{P}/\mathfrak{p}} = 1$ but also that the residue field extension is separable. If one left out the second condition, the above example would be an unramified extension which becomes ramified after the base change from $\mathbb{F}_p(t)$ to $\mathbb{F}_p(s)$, which is just one of the things that would go wrong.

2. Consider a Dedekind ring A with quotient field K, a finite Galois extension L/K, and let B denote the integral closure of A in L. Consider a subextension K'/K which is also Galois and let A' denote the integral closure of A in K'. Consider a prime $\mathfrak{p}$ of A and a prime $\mathfrak{P} \subset B$ above $\mathfrak{p}$, such that $k(\mathfrak{P})/k(\mathfrak{p})$ is separable. Determine the decomposition of $\mathfrak{p}$ in A' with its numerical invariants r, e, f and its decomposition and inertia groups from the corresponding data in B.

3. Consider a Dedekind ring A with quotient field K, a finite separable extension L/K, and let B denote the integral closure of A in L. Assume that $L = K(\alpha)$, where the minimal polynomial $f(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ of α over K lies in $A[X]$ and is *Eisenstein at* a prime ideal $\mathfrak{p}$ of A, that is, all $a_i \in \mathfrak{p}$ and $a_0 \notin \mathfrak{p}^2$. Show that $\mathfrak{p}B = \mathfrak{P}^n$ with $\mathfrak{P} := \mathfrak{p}B + \alpha B$ prime, so that $\mathfrak{p}$ is totally ramified in B.

   (*Hint:* Prove that $\mathfrak{p}B \subset \mathfrak{P}^j$ for all $j \leqslant n$ by induction on j.)

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4. Let L/K be a Galois extension of number fields with noncyclic Galois group.

   (a) Show that any prime ideal of $\mathcal{O}_K$ over which lies only one prime ideal of $\mathcal{O}_L$ is ramified in $\mathcal{O}_L$.

   (b) Deduce that there are at most finitely many prime ideals with the property in (a), and in particular no prime ideals of $\mathcal{O}_K$ that are totally inert in $\mathcal{O}_L$.

5. For $K := \mathbb{Q}(\sqrt[3]{2})$ compute the prime factorization of the different $\mathrm{diff}_{\mathcal{O}_K/\mathbb{Z}}$ and verify that a prime ideal of $\mathcal{O}_K$ divides $\mathrm{diff}_{\mathcal{O}_K/\mathbb{Z}}$ if and only if it is ramified over $\mathbb{Z}$.