Exercise Sheet 7

Different and Discriminant, Cyclotomic Fields

1. (a) Prove that any Dedekind ring with only finitely many maximal ideals is a principal ideal domain.

   (b) Let A be a discrete valuation ring and B its integral closure in a finite separable field extension of $\mathrm{Quot}(A)$. Deduce from (a) that B is a principal ideal domain.

2. Let $K := \mathbb{Q}(\alpha)$, where $\alpha := \sqrt[3]{539}$.

   (a) Using exercise 3 of sheet 6, show that (7) and (11) are totally ramified in $\mathcal{O}_K$. Let $\mathfrak{p}_7$ and $\mathfrak{p}_{11}$ denote the prime ideals above (7) and (11), respectively.

   (b) Using the discriminant, show that $\mathcal{O}_K = \alpha\mathbb{Z} \oplus \beta\mathbb{Z} \oplus \gamma\mathbb{Z}$, where $\beta := \frac{77}{\alpha}$ and $\gamma := \frac{1+2\alpha+\beta}{3}$, and that $\mathrm{disc}(\mathcal{O}_K) = -3 \cdot 7^2 \cdot 11^2$.

   (c) Show that $3\mathcal{O}_K = \mathfrak{p}_3^2\mathfrak{p}_3'$ for distinct prime ideals $\mathfrak{p}_3$ and $\mathfrak{p}_3'$.

   (d) Show that the different of $\mathcal{O}_K/\mathbb{Z}$ is $\mathfrak{p}_3\mathfrak{p}_7^2\mathfrak{p}_{11}^2$.

   *(e) Using the norm, show that $\mathrm{diff}_{\mathcal{O}_K/\mathbb{Z}}$ is not principal and conclude that $\mathcal{O}_K$ is not generated by one element over $\mathbb{Z}$.

3. Let K be a number field, let m be a positive integer, let $G_m(K) := \{x^m \mid x \in K^\times\}$ and let $L_m(K)$ be the group of elements $x \in K^\times$ such that in the prime factorization of (x), all exponents are multiples of m.

   (a) Prove that for every $x \in L_m(K)$, there exists a unique fractional ideal $\mathfrak{a}_x$ such that $(x) = \mathfrak{a}_x^m$.

   (b) Define $S_m(K) := L_m(K)/G_m(K)$ and $\mathrm{Cl}(\mathcal{O}_K)[m] := \{c \in \mathrm{Cl}(\mathcal{O}_K) \mid c^m = 1\}$ and show that the map

$$\begin{aligned} f : S_m(K) &\to \mathrm{Cl}(\mathcal{O}_K)[m] \\ x &\mapsto [\mathfrak{a}_x] \end{aligned}$$

   is a well-defined group homomorphism.

   (c) Show that f is surjective.

   (d) Find the kernel of f.

*4. (*Hilbert's Theorem 90*) Let L/K be a finite Galois extension of fields whose Galois group is cyclic and generated by σ. Show that for any element $x \in L^\times$ with $\mathrm{Nm}_{L/K}(x) = 1$ there exists an element $y \in L^\times$ with $x = \sigma(y)/y$.

Hint: Set $n := [L/K]$ and consider the map

$$h \colon L \longrightarrow L, \quad z \mapsto h(z) := \sum_{i=0}^{n-1} \sigma^i(z) \cdot \prod_{i<j<n} \sigma^j(x).$$

*5. Set $d := p_1 \cdots p_r$ for prime numbers $2 = p_1 < p_2 < \ldots < p_r$ and consider the imaginary quadratic number field $K := \mathbb{Q}(\sqrt{-d})$. For each i write $p_i \mathcal{O}_K = \mathfrak{p}_i^2$. Show that the subgroup $H := \{\xi \in \mathrm{Cl}(\mathcal{O}_K) \mid \xi^2 = 1\}$ has order 2^{r-1} and is generated by the ideal classes $[\mathfrak{p}_i]$ with the single relation $[\mathfrak{p}_1] \cdots [\mathfrak{p}_r] = 1$.

6. Show that for any root of unity $\zeta \in \mathbb{C}$ whose order is not a prime power, the element $1 - \zeta$ is a unit in $\mathcal{O}_{\mathbb{Q}(\zeta)}$.