Exercise Sheet 9

p-adic Numbers, Absolute Values

1. Determine the p-adic expansions of ± 1 and $\frac{\pm 1}{1-p}$ for an arbitrary prime p.

2. Represent the rational numbers $\frac{2}{3}$ and $-\frac{2}{3}$ as 5-adic numbers.

3. (a) Show that a rational number x with $\mathrm{ord}_p(x) = 0$ has a purely periodic p-adic expansion if and only if $x \in [-1, 0)$.

   (b) Show that in $\mathbb{Q}_p$ the numbers with eventually periodic p-adic expansions are precisely the rational numbers.

4. Show that the equation $x^2 = 2$ has a solution in $\mathbb{Z}_7$ and compute its first few 7-adic digits.

*5. For any integer $n \geqslant 2$ consider the map

$$\pi \colon \prod_{i \geqslant 1} \{0, 1, \ldots, n-1\} \longrightarrow [0, 1], \quad (a_i)_i \mapsto \sum_{i \geqslant 1} a_i n^{-i}.$$

Show that π is surjective and determine its fibers. Prove that the natural topology on the interval $[0, 1]$ is the quotient topology via π from the product topology on $\prod_{i \geqslant 1} \{0, 1, \ldots, n-1\}$, where each factor is endowed with the discrete topology. Interpret this fact by comparing the topologies on the source and the target.

6. Consider the sequence of integers defined by $a_1 := 5$ and $a_{i+1} := a_i^2$ for all $i \geqslant 1$. Write the decimal expansions of these a_i below each other. Observe the pattern and formulate and prove a theorem about it. Explain the pattern by comparison with p-adic numbers. Does a similar pattern occur with other starting values and other bases besides 10 for the expansion?

7. Let $|\cdot|$ be an absolute value on a field K. Show that $|\cdot|^\alpha$ is also an absolute value for every $0 < \alpha \leqslant 1$.