D-MATH | Algebraic Number Theory | FS 2018
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Exercise Sheet 10

p-adic Numbers, Absolute Values, Completion

1. Let p be a prime number.
   (a) Show that the sequence $\frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \ldots$ does not converge in $\mathbb{Q}_p$.
   (b) For any $a \in \mathbb{Z}$ coprime to p show that the sequence $(a^{p^n})_{n\geqslant 1}$ converges in $\mathbb{Q}_p$.
   (c) Determine this limit.

2. Here we consider $\mathbb{Q}_p$ as an abstract field and include $\mathbb{Q}_\infty := \mathbb{R}$.
   (a) Show that $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic for any $p \neq q$.
   (b) Prove that every automorphism of $\mathbb{Q}_p$ is trivial.

   Hint: Look at which integers are squares in the respective field.

*3. Show that there is a canonical isomorphism $\mathbb{Z}[[X]]/(X-p) \xrightarrow{\sim} \mathbb{Z}_p$.

4. Show that for any absolute value $|\ |$ on a field K, the maps $+, \cdot \colon K \times K \to K$ and $(\ )^{-1} \colon K \smallsetminus \{0\} \to K \smallsetminus \{0\}$ are continuous for the induced topology.

5. Let K be a complete non-archimedean field. Show that a series $\sum_{n=0}^{\infty} a_n$ with summands in K converges if and only if $\lim\limits_{n\to\infty} a_n = 0$ in K.

6. Let K be a field that is complete with respect to a p-adic absolute value. Consider $x \in K$ with $|x| < 1$ and $\alpha, \beta \in \mathbb{Z}_p$ and $m, n \in \mathbb{Z}$ with $n \geqslant 0$. Prove:
   (a) The binomial coefficient $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ lies in $\mathbb{Z}_p$.
   (b) $F_\alpha(x) := \sum_{n\geqslant 0} \binom{\alpha}{n} x^n \in K$ is well-defined and satisfies $|F_\alpha(x) - 1| < 1$.
   (c) $F_{\alpha+\beta}(x) = F_\alpha(x) \cdot F_\beta(x)$.
   (d) $F_{m\alpha}(x) = F_\alpha(x)^m$.
   (e) $F_m(x) = (1+x)^m$.
   (f) $y := F_{m/n}(x)$ is the only solution of the equation $y^n = (1+x)^m$ with $|y-1| < 1$, if $p \nmid n$.

   This therefore justifies writing $F_\alpha(x) = (1+x)^\alpha$.

   *(g) Do we then also have $((1+x)^\alpha)^\beta = (1+x)^{\alpha\beta}$?

*7. *(Newton method for finding zeros of a polynomial)* Let p be a prime number, let $f \in \mathbb{Z}_p[X]$ and let $\alpha \in \mathbb{Z}_p$ be a root of f such that $f'(\alpha) \neq 0$. Set

$$U := \{a \in \mathbb{Z}_p \mid |f(a)| < |f'(a)|^2 \text{ and } |\alpha - a| < |f'(a)|\},$$

which is an open neighborhood of α in $\mathbb{Z}_p$. Let $a_1 \in U$ and recursively define $a_{n+1} := a_n - \frac{f(a_n)}{f'(a_n)}$ for $n \geqslant 1$. Show that for all n:

(a) $a_n \in U$,

(b) $|f'(a_n)| = |f'(a_1)|$,

(c) $|f(a_n)| \leqslant |f'(a_1)|^2 t^{2^{n-1}}$ for $t = |f(a_1)/f'(a_1)| < 1$.

Moreover, show that $\lim_{n\to\infty} a_n = \alpha$ and $|f'(\alpha)| = |f'(a_1)|$.