Exercise Sheet 12

Extensions of Absolute Values, Local and Global Fields

1. Let L/K be a purely inseparable finite extension of degree q. Show that every absolute value $|\ |$ on K possesses a unique extension to L, given by the formula
$$|y| \ := \ |y^q|^{q^{-1}}.$$

*2. Let L/K be a finite field extension and let $|\ |$ be a (nontrivial) absolute value on L. Show that the restriction of $|\ |$ to K is nontrivial.

(*Hint:* Use Newton polygons.)

3. (a) Determine all the absolute values on $\mathbb{Q}(\sqrt{5}\,)$.
   (b) How many extensions to $\mathbb{Q}(\sqrt[n]{2}\,)$ does the archimedean absolute value on $\mathbb{Q}$ admit?

4. Let p be a prime number and $\bar{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$.
   (a) Show that $|\ |_p$ extends to some absolute value $|\ |$ on $\bar{\mathbb{Q}}$.
   (b) For any subfield $K \subset \bar{\mathbb{Q}}$ which is finite over $\mathbb{Q}$ let $\hat{K}$ be the completion of K with respect to the restriction of $|\ |$. Show that for any subfields $K \subset L \subset \bar{\mathbb{Q}}$ which are finite over $\mathbb{Q}$ we get a natural inclusion $\hat{K} \hookrightarrow \hat{L}$.
   (c) Show that the union $\bar{\mathbb{Q}}_p$ of all these $\hat{K}$ is an algebraic closure of $\mathbb{Q}_p$.
   (d) Show that there is a natural isomorphism
$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Stab}_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}(|\ |).$$

5. *(Product formula)* A non-archimedean absolute value $|\ |$ on a field K for which $\mathcal{O}_K$ is a discrete valuation ring with finite residue field $\mathcal{O}_K/\mathfrak{m}$ is called *normalized* if $|\pi| = |\mathcal{O}_K/\mathfrak{m}|^{-1}$ for any element π with $(\pi) = \mathfrak{m}$. The usual absolute value on $\mathbb{Q}_p$ is normalized.
   (a) Show that for all $a \in \mathbb{Q}^\times$ we have $\prod_{p\leqslant\infty} |a|_p = 1$.
   (b) For any finite field k, write down all normalized absolute values on $k(t)$.
   (c) For any finite field k and any $a \in k(t)^\times$, prove that $\prod_v |a|_v = 1$, where the product is taken over all normalized absolute values on $k(t)$.

*6. Show that any local field is the completion of a global field at an absolute value.