Exercise Sheet 13

Zeta Functions

1. Consider real numbers $1 < a_1 < a_2 < \ldots$ with $\sum_{k=1}^{\infty} a_k^{-1} = \infty$. For any integer n let α_n denote the number of $k \geqslant 1$ with $a_k \leqslant n$. Prove that for every $\varepsilon > 0$

   (a) there exist infinitely many k with $a_k \leqslant \varepsilon k (\log k)^{1+\varepsilon}$.

   (b) there exist infinitely many n with $\alpha_n \geqslant \frac{n}{\varepsilon (\log n)^{1+\varepsilon}}$.

   *(c) Suppose that $a_k = k(\log k)^c$ for some constant $c \geqslant 0$. Determine the asymptotic behavior of $\sum a_k^{-s}$ for real $s \to 1+$.

2. Show that for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we have

   (a)
   $$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$
   where μ denotes the Möbius function.

   (b)
   $$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$
   where $d(n)$ is the number of prime divisors of n.

   (c)
   $$\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\log p}{p^{ns}}.$$

   *(d)
   $$\log \zeta(s) = s \int_2^{\infty} \frac{\pi(x)}{x(x^s - 1)}\, dx,$$
   where $\pi(x)$ denotes the number of primes $p \leqslant x$.

3. Let $\mathbb{F}_q$ denote a finite field of cardinality q, and consider a ring of the form $A := \mathbb{F}_q[X_1, \ldots, X_r]/(f_1, \ldots, f_s)$ for polynomials $f_1, \ldots, f_s$. For every ideal $\mathfrak{a} \subset A$ of finite index set $\deg(\mathfrak{a}) := \dim_{\mathbb{F}_q}(A/\mathfrak{a})$. The *formal zeta function* of A is the formal power series

$$Z(T) \;:=\; \prod_{\mathfrak{m} \subset A} (1 - T^{\deg(\mathfrak{m})})^{-1} \;\in\; \mathbb{Z}[[T]]^\times,$$

where the product is extended over all maximal ideals $\mathfrak{m} \subset A$. For any integer $n \geqslant 1$ let $\mathbb{F}_{q^n}$ be an extension of degree n and put

$$X(\mathbb{F}_{q^n}) \;:=\; \big\{ \underline{x} \in (\mathbb{F}_{q^n})^r \;\big|\; f_1(\underline{x}) = \ldots = f_s(\underline{x}) = 0 \big\}.$$

(*Explanation:* Here X denotes the affine algebraic variety over $\mathbb{F}_q$ defined by the equations $f_1 = \ldots = f_s = 0$, and A is its coordinate ring.)

(a) Prove that $Z(T)$ is well-defined and satisfies

$$T \frac{d}{dT} \log Z(T) \;=\; T \frac{Z'(T)}{Z(T)} \;=\; \sum_{n \geqslant 1} |X(\mathbb{F}_{q^n})| \cdot T^n.$$

(b) If A is a Dedekind ring prove that

$$Z(T) \;=\; \sum_{0 \neq \mathfrak{a} \subset A} T^{\deg(\mathfrak{a})}.$$

(c) In the case $A := \mathbb{F}_q[X_1, \ldots, X_r]$ prove that

$$Z(T) \;=\; (1 - q^r T)^{-1}.$$

(d) Prove that the number N_d of monic irreducible polynomials of degree d in $\mathbb{F}_q[X]$ satisfies

$$N_d \;=\; \tfrac{1}{d} \cdot \sum_{k|d} \mu(\tfrac{d}{k}) q^k,$$

where μ is the Möbius function.