D-MATH | Algebraic Number Theory | FS 2018
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Exercise Sheet 14

Dirichlet Density, Primes in Arithmetic Progressions

1. Does there exist a number field which does not embed into $\mathbb{Q}_p$ for any p?

2. Determine the Dirichlet density of the set of primes $p \equiv 3 \bmod(4)$ that split completely in the field $\mathbb{Q}(\sqrt[3]{2})$.

3. Let L/K be an extension of number fields. Prove that $L = K$ if and only if the set of primes $\mathfrak{p} \subset \mathcal{O}_K$ which are totally split in L has Dirichlet density $> \frac{1}{2}$.

4. Let L/K be an extension of number fields. Prove that L/K is galois if and only if for almost all primes $\mathfrak{p} \subset \mathcal{O}_K$, if there exists a prime $\mathfrak{P}|\mathfrak{p}$ of $\mathcal{O}_L$ with $f_{\mathfrak{P}/\mathfrak{p}} = 1$, then $\mathfrak{p}$ is totally split in $\mathcal{O}_L$.

5. Let a be an integer that is not a third power. Let A be the set of prime numbers p such that $a \bmod (p)$ is a third power in $\mathbb{F}_p$.

   (a) Prove that A and its complement are both infinite.

   (b) Prove that there is no integer N such that the property $p \in A$ depends only on the residue class of p modulo (N).