Solutions 5

Units, Decomposition Of Prime Ideals

1. (a) Determine the ring of integers of $K := \mathbb{Q}(\sqrt{5}, i)$.
   (b) Determine $\mathcal{O}_F^\times$ for the subfield $F := \mathbb{Q}(\sqrt{5})$.
   (c) Find a fundamental unit of $\mathcal{O}_K^\times$.
   (d) Show that $|\mu(K)| = 4$ and write down $\mathcal{O}_K^\times$.

Solution:

(a) Consider the subfields $F := \mathbb{Q}(\sqrt{5})$ and $F' := \mathbb{Q}(i) = \mathbb{Q}(\sqrt{-1})$. Since $5 \equiv 1 \bmod 4$ and $-1 \not\equiv 1 \bmod 4$, their discriminants are $\mathrm{disc}(\mathcal{O}_F) = 5$ and $\mathrm{disc}(\mathcal{O}_{F'}) = -4$ and hence coprime. Furthermore, the fields F and F' are linearly disjoint, since $[FF'/\mathbb{Q}] = [K/\mathbb{Q}] = 4 = [F/\mathbb{Q}] \cdot [F'/\mathbb{Q}]$. Therefore §1 Theorem 22 implies that $\mathcal{O}_K \cong \mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{F'} \cong \mathbb{Z}[\frac{1+\sqrt{5}}{2}, i]$. In particular a $\mathbb{Z}$-basis of $\mathcal{O}_K$ is $1, \frac{1+\sqrt{5}}{2}, i, i\frac{1+\sqrt{5}}{2}$.

(b) By §5 Proposition 15, the element $\varepsilon := a + b\sqrt{5} \in \mathcal{O}_F$ with minimal $a, b \in \frac{1}{2}\mathbb{Z}^{>0}$ such that $\mathrm{Nm}_{F/\mathbb{Q}}(\varepsilon) = \pm 1$ is a fundamental unit in $\mathcal{O}_F^\times$. By a direct calculation, we verify that $\varepsilon := \frac{1+\sqrt{5}}{2}$ already has norm -1 and hence is a fundamental unit. It follows that $\mathcal{O}_F^\times = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$.

(c) The field K has $(r, s) = (0, 2)$ and hence $\mathcal{O}_K^\times = \mu(K) \times \tilde{\varepsilon}^{\mathbb{Z}}$ for some fundamental unit $\tilde{\varepsilon} \in \mathcal{O}_K^\times$. In view of (b) it follows that $\zeta\tilde{\varepsilon}^n = \varepsilon^{\pm 1}$ for some $n \geqslant 1$ and $\zeta \in \mu(K)$. After possibly replacing $\tilde{\varepsilon}$ with $\tilde{\varepsilon}^{-1}$ and ζ with ζ^{-1}, we may assume that $\zeta\tilde{\varepsilon}^n = \varepsilon$. Writing $\mathrm{Nm}_{K/F}(\tilde{\varepsilon}) = \pm\varepsilon^k$ with $k \in \mathbb{Z}$, we deduce that

$$\varepsilon^2 = \mathrm{Nm}_{K/F}(\varepsilon) = \mathrm{Nm}_{K/F}(\zeta\tilde{\varepsilon}^n) = \pm \mathrm{Nm}_{K/F}(\tilde{\varepsilon})^n = \pm(\pm\varepsilon^k)^n,$$

which implies that $kn = 2$. Suppose that $n = 2$ and hence $k = 1$. Write $\tilde{\varepsilon} = a + b\frac{1+\sqrt{5}}{2} + ci + di\frac{1+\sqrt{5}}{2}$ with $a, b, c, d \in \mathbb{Z}$. Then

$$\pm\frac{1+\sqrt{5}}{2} = \pm\varepsilon = \mathrm{Nm}_{K/F}(\tilde{\varepsilon}) = \tilde{\varepsilon}\bar{\tilde{\varepsilon}} = (a^2+b^2+c^2+d^2)+(2ab+b^2+2cd+d^2)\frac{1+\sqrt{5}}{2}.$$

Comparing coefficients implies that $a^2 + b^2 + c^2 + d^2 = 0$ and hence $a = b = c = d = 0$. This contradicts the fact that $\tilde{\varepsilon} \neq 0$. Therefore $n = 1$ and $\tilde{\varepsilon} = \zeta^{-1}\varepsilon$ is also a fundamental unit in $\mathcal{O}_K^\times$. Since the fundamental unit of K is only determined up multiplication with an element of $\mu(K)$ and taking its inverse, we conclude that ε is a fundamental unit in $\mathcal{O}_K^\times$.

(d) Let ζ be a generator of $\mu(K)$ and let n be the order of ζ. Then $[\mathbb{Q}(\zeta)/\mathbb{Q}] = \varphi(n)$, where $\varphi(\cdot)$ denotes the Euler φ-function, and this divides $[K/\mathbb{Q}] = 4$. On the other hand, since $i \in K$, we have $n = 2^k m$ with m odd and $k \geqslant 2$ and hence $\varphi(n) = (2^k - 2^{k-1})\varphi(m) = 2^{k-1}\varphi(m)$. Together this leaves only the possibilities $n = 4$, 8, 12.
If $n = 8$, we have $\zeta = \frac{\pm 1 \pm i}{\sqrt{2}}$ and hence $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\zeta + \bar{\zeta}) \subset K$.
If $n = 12$, we have $\zeta^4 = \frac{-1 \pm \sqrt{-3}}{2}$ and hence $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta^4) \subset K$.
But the extension $K/\mathbb{Q}$ is galois with a non-cyclic Galois group of order 4; hence by Galois theory it contains precisely three different quadratic subfields. Since $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(i\sqrt{5}) = \mathbb{Q}(\sqrt{-5})$ are all contained in K and non-isomorphic by the classification of quadratic number fields, these are precisely all quadratic subfields of K. Again by the classification of quadratic number fields, none of them is isomorphic to $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-3})$. Thus the cases $n = 8$, 12 are impossible, leaving only $n = 4$.
In conclusion, we have $|\mu(K)| = 4$ and $\mathcal{O}_K^\times = \{\pm 1, \pm i\} \times (\frac{1+\sqrt{5}}{2})^{\mathbb{Z}}$.

2. (a) Let K be a cubic number field with exactly one real embedding. We identify K with its image. Show that for any unit $u \in \mathcal{O}_K^\times$ with $u > 1$ we have

$$|\operatorname{disc}(\mathcal{O}_K)| \leqslant 3\left(u^2 + \frac{2}{u}\right)\left(u^4 + \frac{2}{u^2}\right).$$

Hint: Use Hadamard's inequality: For any complex $n \times n$-matrix M with columns $v_1, \ldots, v_n$, we have $|\det(M)| \leqslant \prod_{i=1}^n \|v_i\|$.

(b) Show that a fundamental unit of $\mathcal{O}_K^\times$ for the number field $K := \mathbb{Q}(\sqrt[3]{2})$ is $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

Solution: (a) Let $\sigma : K \to \mathbb{C}$ be any nonreal embedding, and take any unit $u > 1$ in $\mathcal{O}_K$. Then $u \notin \mathbb{Q}$ and hence $K = \mathbb{Q}(u)$. Thus $1, u, u^2$ is a $\mathbb{Q}$-basis of K that is contained in $\mathcal{O}_K$, and so

$$|\operatorname{disc}(\mathcal{O}_K)| \;\leqslant\; \left|\operatorname{disc}(1, u, u^2)\right| \;=\; \left|\det\begin{pmatrix} 1 & u & u^2 \\ 1 & \sigma(u) & \sigma(u)^2 \\ 1 & \bar{\sigma}(u) & \bar{\sigma}(u)^2 \end{pmatrix}\right|^2$$

Since $u|\sigma(u)|^2 = u\sigma(u)\bar{\sigma}(u) = \mathrm{Nm}_{K/\mathbb{Q}}(u) = \pm 1$ and $u > 0$, we have $|\sigma(u)|^2 = |\bar{\sigma}(u)|^2 = \frac{1}{u}$. Using Hadamard's inequality we deduce that

$$|\operatorname{disc}(\mathcal{O}_K)| \;\leqslant\; \left\|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\|^2 \cdot \left\|\begin{pmatrix} u \\ \sigma(u) \\ \bar{\sigma}(u) \end{pmatrix}\right\|^2 \cdot \left\|\begin{pmatrix} u^2 \\ \sigma(u)^2 \\ \bar{\sigma}(u)^2 \end{pmatrix}\right\|^2 \;\leqslant\; 3\left(u^2 + \frac{2}{u}\right)\left(u^4 + \frac{2}{u^2}\right).$$

(b) From the solution of exercise 3 on sheet 1 we know that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$ with discriminant -108 and that the norm is given by the formula

$$\mathrm{Nm}_{K/\mathbb{Q}}\big(a + b\sqrt[3]{2} + c\sqrt[3]{4}\big) \;=\; 6abc - a^3 - 2ab^3 - 4ac^3$$

for any $a, b, c \in \mathbb{Z}$. In particular the element $\alpha := 1 + \sqrt[3]{2} + \sqrt[3]{4}$ has norm $\mathrm{Nm}_{K/\mathbb{Q}}(\alpha) = -1 \in \mathbb{Z}^\times$ and is therefore a unit. Since K has $(r, s) = (1, 1)$, by Dirichlet's unit theorem we have $\mathcal{O}_K^\times = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$ for a fundamental unit ε. After possibly replacing ε by $\pm\varepsilon^{\pm 1}$, we may assume that $\varepsilon > 1$. Since $\alpha > 1$, it then follows that $\alpha = \varepsilon^k$ for some integer $k \geqslant 1$. We must show that $k = 1$.

For this let $\lambda(u)$ denote the right-hand side of the inequality in part (a). Looking at the derivative shows that this is a monotone increasing function of $u \geqslant 1$. Thus if $k \geqslant 3$, it follows that

$$108 \leqslant \lambda(\varepsilon) = \lambda(\sqrt[k]{\alpha}) \leqslant \lambda(\sqrt[3]{\alpha}) = 76.6\ldots$$

which is a contradiction. Therefore $k \leqslant 2$. To rule out $k = 2$ we must show that α is not a square in $\mathcal{O}_K$. For this observe that $\mathfrak{p} := (5, 2 + \sqrt[3]{2})$ is a prime ideal with $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_5$ and $\alpha + \mathfrak{p} = 3 + \mathfrak{p}$. Since 3 is not a square in $\mathbb{F}_5$, it follows that α is not a square in $\mathcal{O}_K$. Hence $k = 1$ and α is a fundamental unit.

3. Using continued fractions:

   (a) Compute a fundamental unit of $\mathcal{O}_K^\times$ for $K := \mathbb{Q}(\sqrt{318})$.

   (b) Find the smallest positive integer solution of the equation $x^2 - 61y^2 = 1$.

Solution: (a) Because $318 \not\equiv 1 \bmod 4$, we have $\mathcal{O}_K = \mathbb{Z}[\sqrt{318}]$. Let $(\ \ )'$ denote the nontrivial Galois automorphism of K over $\mathbb{Q}$. Since $\sqrt{318} = 17.8\ldots$, the element $\eta := 17 + \sqrt{318}$ satisfies $\mathcal{O}_K = \mathbb{Z}[\eta]$ with $\eta > 1$ and $-1 < \eta' < 0$. The continued fraction expansion of η is obtained by the calculation

$$\begin{aligned}
\eta_0 &:= \eta \\
a_0 &:= \lfloor \eta_0 \rfloor = 34 \\
\eta_1 &:= \frac{1}{\eta_0 - a_0} = \frac{1}{-17 + \sqrt{318}} = \frac{-17 - \sqrt{318}}{17^2 - 318} = \frac{\eta}{29} = 1.2\ldots \\
a_1 &:= 1 \\
\eta_2 &:= \frac{1}{\eta_1 - a_1} = \frac{1}{\frac{\eta}{29} - 1} = \frac{29(-12 - \sqrt{318})}{12^2 - 318} = 2 + \frac{\sqrt{318}}{6} = 4.97\ldots \\
a_2 &:= 4 \\
\eta_3 &:= \frac{1}{\eta_2 - a_2} = \frac{1}{-2 + \frac{\sqrt{318}}{6}} = \frac{6(-6 - \sqrt{318})}{12^2 - 318} = \frac{12 + \sqrt{318}}{29} = 1.02\ldots \\
a_3 &:= 1 \\
\eta_4 &:= \frac{1}{\eta_3 - a_3} = \frac{1}{\frac{-17 + \sqrt{318}}{29}} = \frac{29(-17 - \sqrt{318})}{17^2 - 318} = \eta
\end{aligned}$$

Here the sequence $(a_i) = (\overline{34, 1, 4, 1})$ is periodic with period 4. We further calculate the numerators and denominators of the approximations to η:

i	-2	-1	0	1	2	3
a_i			34	1	4	1
p_i	0	1	34	35	174	209
q_i	1	0	1	1	5	6

Thus a fundamental unit in $\mathcal{O}_K^\times$ is

$$\varepsilon := p_3 - q_3\eta = 209 - 6(17 + \sqrt{318}) = 107 - 6\sqrt{318}.$$

(b) This is a challenge that Pierre de Fermat sent to some fellow mathematicians in 1657 and which was solved by several of them. Our source for the question is [J. S. Silverman: A friendly introduction to number theory, 4th Ed., Pearson 2013] Chapter 32.

Let $K := \mathbb{Q}(\sqrt{61})$. Then $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{61}}{2}]$, because $61 \equiv 1 \bmod 4$. Let ε be the fundamental unit of K in $\mathbb{R}^{>1}$. We are looking for the smallest power $\varepsilon^k = a + b\sqrt{61}$ that satisfies $\mathrm{Nm}_{K/\mathbb{Q}}(\varepsilon^k) = a^2 - 61b^2 = 1$ with $a, b \in \mathbb{Z}$.

Let $(\ )'$ denote the nontrivial Galois automorphism of K over $\mathbb{Q}$. Since $\left(\frac{1+\sqrt{61}}{2}\right)' = \frac{1-\sqrt{61}}{2} = -3.4\ldots$, the element $\eta := \frac{7+\sqrt{61}}{2}$ satisfies $\mathcal{O}_K = \mathbb{Z}[\eta]$ with $\eta > 1$ and $-1 < \eta' < 0$. The continued fraction expansion of η is obtained by the calculation

$$\begin{aligned}
\eta_0 &:= \eta = 7.4\ldots \\
a_0 &:= \lfloor \eta \rfloor = 7 \\
\eta_1 &:= \frac{1}{\eta_0 - a_0} = \frac{1}{\frac{-7+\sqrt{61}}{2}} = \frac{2(-7-\sqrt{61})}{7^2 - 61} = \frac{7+\sqrt{61}}{6} = 2.4\ldots \\
a_1 &:= 2 \\
\eta_2 &:= \frac{1}{\eta_1 - a_1} = \frac{1}{\frac{-5+\sqrt{61}}{6}} = \frac{5+\sqrt{61}}{6} = 2.1\ldots \\
a_2 &:= 2 \\
\eta_3 &:= \frac{1}{\eta_2 - a_2} = \frac{1}{\frac{-7+\sqrt{61}}{6}} = \eta
\end{aligned}$$

Here the sequence $(a_i) = (\overline{7, 2, 2})$ is periodic with period 3. We calculate the numerators and denominators of the approximations to η:

i	-2	1	0	1	2
a_i			7	2	2
p_i	0	1	7	15	37
q_i	1	0	1	2	5

Hence a fundamental unit in $\mathcal{O}_K^\times$ is

$$\tilde{\varepsilon} := p_2 - q_2\eta = 37 - 5\tfrac{7+\sqrt{61}}{2} = \tfrac{39-5\sqrt{61}}{2}.$$

The corresponding fundamental unit $\varepsilon > 1$ is therefore $\varepsilon := \tilde{\varepsilon}' = \frac{39+5\sqrt{61}}{2}$. Since $\mathrm{Nm}_{K/\mathbb{Q}}(\varepsilon) = (39^2 - 5^2 \cdot 61)/4 = -1$, our desired exponent k must be even. Since $\varepsilon^2 = \frac{1523+195\sqrt{61}}{2}$ does not yield an integer solution to the equation, we compute:

$$\varepsilon^4 = \frac{2319527 + 296985\sqrt{61}}{2}$$
$$\varepsilon^6 = 1766319049 + 226153980\sqrt{61}$$

The answer is therefore $(x, y) = (1766319049, 226153980)$.

4. Let K be a number field and let S be a finite set of prime ideals of $\mathcal{O}_K$. We define the ring of *S-integers in K* to be

$$\mathcal{O}_{K,S} \;:=\; \bigcap_{\mathfrak{p}\notin S} \mathcal{O}_{K,\mathfrak{p}} \;=\; \big\{\alpha \in K \;\big|\; \forall \mathfrak{p} \notin S : \mathrm{ord}_{\mathfrak{p}}(\alpha) \geqslant 0\big\}.$$

The group $\mathcal{O}_{K,S}^\times$ is called the group of *S-units in K*.

(a) Show that the torsion subgroup of $\mathcal{O}_{K,S}^\times$ is $\mu(K)$.

(b) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the distinct elements of S. Show that the homomorphism

$$\varphi\colon \mathcal{O}_{K,S}^\times \to \mathbb{Z}^t,\; \alpha \mapsto (\mathrm{ord}_{\mathfrak{p}_i}(\alpha))_i$$

has kernel $\mathcal{O}_K^\times$ and that its image has rank t.

(c) Deduce that $\mathcal{O}_{K,S}^\times \cong \mu(K) \times \mathbb{Z}^{r+s+|S|-1}$.

Solution: This proof partly follows Milne's notes on algebraic number theory, page 89: `http://www.jmilne.org/math/CourseNotes/ANT.pdf`.

(a) Since the torsion subgroup of $K^\times$ is $\mu(K)$, the torsion subgroup of $\mathcal{O}_{K,S}^\times$ must be a subgroup of $\mu(K)$. But $\mu(K) \subseteq \mathcal{O}_K^\times \subseteq \mathcal{O}_{K,S}^\times$ and the conclusion follows.

(b) For each non-zero prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ the localization $\mathcal{O}_{K,\mathfrak{p}}$ is a discrete valuation ring; hence $\mathcal{O}_{K,\mathfrak{p}}^\times = \{\alpha \in K \mid \mathrm{ord}_{\mathfrak{p}}(\alpha) = 0\}$. Taking the intersection over all $\mathfrak{p} \notin S$, it follows that

$$\mathcal{O}_{K,S}^\times \;=\; \bigcap_{\mathfrak{p}\notin S} \mathcal{O}_{K,\mathfrak{p}}^\times \;=\; \big\{\alpha \in K \;\big|\; \forall \mathfrak{p} \notin S : \mathrm{ord}_{\mathfrak{p}}(\alpha) = 0\big\}.$$

Taking the intersection over all $\mathfrak{p}$, it follows that

$$\mathcal{O}_K^\times \;=\; \big\{\alpha \in \mathcal{O}_{K,S}^\times \;\big|\; \forall \mathfrak{p} \in S : \mathrm{ord}_{\mathfrak{p}}(\alpha) = 0\big\} \;=\; \mathrm{Ker}(\varphi).$$

Let h be the class number of $\mathcal{O}_K$. Then each $\mathfrak{p}_i^h = (\pi_i)$ for an element $\pi_i \in \mathcal{O}_{K,S}^\times$ with

$$\varphi(\pi_i) = (0, \dots, h, \dots, 0).$$

Thus $h\mathbb{Z}^t \subset \mathrm{Im}(\varphi) \subset \mathbb{Z}^t$, and so $\mathrm{Im}(\varphi)$ is a free abelian group of rank $t = |S|$.

(c) By (b), we obtain a short exact sequence

$$1 \to \mathcal{O}_K^\times \to \mathcal{O}_{K,S}^\times \to \mathrm{Im}(\varphi) \cong \mathbb{Z}^t \to 0.$$

Since $\mathrm{Im}(\varphi)$ is free of rank t the sequence splits and we have $\mathcal{O}_{K,S}^\times \cong \mathcal{O}_K^\times \times \mathbb{Z}^t \cong \mu(K) \times \mathbb{Z}^{r+s+t-1}$.

5. In the number field $K := \mathbb{Q}(\sqrt[3]{2})$, what are the possible decompositions of $p\mathcal{O}_K$ for rational primes p?

Solution: Let p be a rational prime and $p\mathcal{O}_K = \prod_{i=1}^r \mathfrak{p}_i^{e_i}$ its prime factorization in $\mathcal{O}_K$. Then $\sum_{i=1}^r e_i f_i = [K/\mathbb{Q}] = 3$. Hence $1 \leqslant r \leqslant 3$ and the possibilities for $(r; e_1, f_1; e_2, f_2; \dots)$ are, up to permutation of the $\mathfrak{p}_i$:

$$\begin{aligned} r = 1: \quad & (1; 3, 1) \\ & (1; 1, 3) \\ r = 2: \quad & (2; 1, 1; 2, 1) \\ & (2; 1, 1; 1, 2) \\ r = 3: \quad & (3; 1, 1; 1, 1; 1, 1) \end{aligned}$$

To compute the decomposition recall from the solution of exercise 3 on sheet 1 that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}] \cong \mathbb{Z}[X]/(X^3 - 2)$. For any prime p we therefore have $\mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{F}_p[X]/(X^3 - 2)$, and the prime factorization of $p\mathcal{O}_K$ corresponds to the prime factorization of $X^3 - 2$ in $\mathbb{F}_p[X]$. For instance

$$\begin{array}{ll} \mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[X]/(X^3) & \rightsquigarrow (1; 3, 1) \\ \mathcal{O}_K/3\mathcal{O}_K \cong \mathbb{F}_3[X]/(X - 2)^3 & \rightsquigarrow (1; 3, 1) \\ \mathcal{O}_K/5\mathcal{O}_K \cong \mathbb{F}_5[X]/((X - 3)(X^2 + 3X + 4)) & \rightsquigarrow (2; 1, 1; 1, 2) \\ \mathcal{O}_K/7\mathcal{O}_K \cong \mathbb{F}_7[X]/(X^3 - 2) & \rightsquigarrow (1; 1, 3) \\ \mathcal{O}_K/31\mathcal{O}_K \cong \mathbb{F}_{31}[X]/((X - 4)(X - 7)(X - 20)) & \rightsquigarrow (3; 1, 1; 1, 1; 1, 1) \end{array}$$

Hence we found all theoretically possible decompositions except $(2; 1, 1; 2, 1)$. We claim that this type does not occur and present two proofs for it:

Using separability of polynomials: If the decomposition $(2; 1, 1; 2, 1)$ occurs for some prime p, then $X^3 - 2 \equiv (X - a)^2(X - b) \bmod p$ for some distinct $a, b \in \mathbb{Z}$. Hence the image of $X^3 - 2$ in $\mathbb{F}_p[X]$ is not separable. In this case, we have for the discriminant Δ of $X^3 - 2$:

$$0 \equiv \Delta = -\det \begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix} = -108 = -2^2 3^3 \bmod p,$$

where the matrix is the Sylvester matrix of $X^3 - 2$ and $\frac{d}{dX}(X^3 - 2) = 3X^2$. Hence $p \in \{2, 3\}$. But in these cases the decomposition type is $(1; 3, 1)$, as shown above. In conclusion, the decomposition cannot be of the form $(2; 1, 1; 2, 1)$.

Using §7 Proposition 13: By §7 Proposition 13, a prime p is ramified if and only if p divides $\operatorname{disc}(\mathcal{O}_K)$. In sheet 1, exercise 3, we calculated $\operatorname{disc}(\mathcal{O}_K) = -108 = -2^2 3^3$. Since 2 and 3 do not ramify with the type $(2; 1, 1; 2, 1)$, no prime decomposes in this way.