D-MATH Algebraic Geometry FS 2018
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Exercise Sheet 1

Classical Varieties

Let K be an algebraically closed field. All algebraic sets below are defined over K, unless specified otherwise.

1. Describe the Zariski topology on $V(XY) \subset \mathbb{A}^2$.

2. Assume that $\mathrm{char}(K) \neq 2, 3$. Show that the polynomial $Y^2 - X^3 - X \in K[X, Y]$ is irreducible. Describe the Zariski topology on $V(Y^2 - X^3 - X) \subset \mathbb{A}^2$.

3. Let $C_1, C_2 \subset \mathbb{A}^2$ be two algebraic curves where C_2 is irreducible. Show that either $C_1 \cap C_2$ is a finite set or $C_2 \subset C_1$.

4. Let $A \subset \mathbb{A}^n$ and $B \subset \mathbb{A}^m$ be two algebraic sets. Prove that their product set $A \times B \subset \mathbb{A}^{n+m}$ is algebraic, too.

5. Consider the group $\mathrm{SL}_n(\mathbb{C})$. Note that it is given by the vanishing set of a polynomial and is thus also an algebraic set in $\mathbb{A}_{\mathbb{C}}^{n^2}$, i.e. an algebraic group.

   Let $H \subset \mathrm{SL}_n(\mathbb{C})$ be a subgroup. Show that the Zariski closure $\overline{H}$ is still a subgroup of $\mathrm{SL}_n(\mathbb{C})$.

   [Hint: observe that the multiplication and inversion maps are morphisms.]

6. Determine the Zariski closure for the following subsets:

   (a) $\{(x, \sin(x)) \mid x \in \mathbb{C}\} \subset \mathbb{A}_{\mathbb{C}}^2$

   (b) $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{C}) \subset \mathbb{A}_{\mathbb{C}}^4$

   [Hint: Consider the subgroups $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{pmatrix}$.]

   (c) $\{(a^2 - b^2, 2ab, a^2 + b^2) \mid a, b \in \mathbb{Z}\} \subset \mathbb{A}_{\mathbb{C}}^3$.

7. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism. This shows that not every morphism whose underlying map of topological spaces is a homeomorphism needs to be an isomorphism.

8. Let $Y \subset \mathbb{A}^3$ be the set $Y := \{(t, t^2, t^3) \mid t \in K\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$ and prove that the coordinate ring $\mathcal{O}(Y)$ is isomorphic to a polynomial ring in one variable over K.

9. *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Show that ψ is well-defined and injective. It is called the *Segre embedding.* Prove that the image of ψ is a projective algebraic set in $\mathbb{P}^N$.

10. Consider the surface Q (i.e. variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw$.

    (a) Show that Q is equal to the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

    (b) Show that Q contains two families of lines (i.e. linear varieties of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the property that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$ then $M_t \cap M_u = \varnothing$ and for all t, u we have $L_t \cap M_u =$ one point.

    (c) Show that Q contains other curves besides these lines and deduce that the Zariski topology on Q is not homeomorphic via the Segre embedding to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$.

11. Let $n, d > 0$ be integers. We denote by $M_0, \dots, M_N$ all monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N := \binom{n+d}{n} - 1$. We define the d-uple embedding as the map $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ sending the point $a = (a_0, \dots, a_n)$ to the point $(M_0(a), \dots, M_N(a))$. Show that the d-uple embedding of $\mathbb{P}^n$ is an isomorphism onto its image.

    [Hint: Look at the inverse map.]