Exercise Sheet 2

Classical Varietes, Rational Maps, Blowups, Spectrum

Let K be an algebraically closed field. All algebraic sets and varieties below are defined over K, unless specified otherwise.

1. Consider the set $M := \mathrm{Mat}_{m,n}(K)$ of $m \times n$-matrices. It can be identified with the affine algebraic variety $\mathbb{A}^{nm}$. Determine if S is open/closed/dense in M:
   (a) $S := \{A \in M \mid A^t A \text{ has an eigenvalue } 1\}$
   (b) $S := \{A \in M \mid \mathrm{rank}(A) = \min\{m, n\}\}$
   (c) for $m = n$, $S := \{A \in M \mid A \text{ is diagonalisable}\}$

2. Construct a morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$ and a closed subvariety $Z \subset \mathbb{A}^2$ such that $f(Z)$ is not closed.

3. Recall the quadric surface Q given by $xy - zw$ in $\mathbb{P}^3$ of exercise 10, sheet 1. Prove that Q is birationally equivalent to $\mathbb{P}^2$.

4. A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona transformation.* Define the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ as $[a_0 : a_1 : a_2] \mapsto [a_1a_2 : a_0a_2 : a_0a_1]$.
   (a) Show that φ is birational, and its own inverse.
   (b) Find open sets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.
   (c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms.

5. *Blowing-up.* We define the *Blowing-up* of $\mathbb{A}^2$ at the point 0 to be the subset $B := \{((x, y), [t : u]) \mid xu = ty\} \subset \mathbb{A}^2 \times \mathbb{P}^1$. Let $\varphi : B \to \mathbb{A}^2$ be the restriction to B of the projection onto the first component (see Figure 1). Prove that:
   (a) The map φ is birational and restricts to an isomorphism $B \setminus \varphi^{-1}(0) \cong \mathbb{A}^2 \setminus 0$.
   (b) We have $\varphi^{-1}(0) \cong \mathbb{P}^1$.
   (c) The points in $\varphi^{-1}(0)$ are in 1-to-1-correspondence with lines ℓ in $\mathbb{A}^2$ through the point 0. [Hint: Look at $\varphi^{-1}(\ell \setminus 0)$ and its closure.]

6. Use the same notation as in the previous exercise. Let C be the curve in $\mathbb{A}^2$ defined by the equation $y^2 = x^2(x + 1)$. Prove that C is singular at the point 0. We define the blowing-up of C to be the closure $\tilde{C} := \overline{\varphi^{-1}(C \setminus 0)}$, see Figure 1. Prove that $\tilde{C}$ is a nonsingular curve. We have thus removed the singularity of C by replacing it with a birationally equivalent curve $\tilde{C}$.

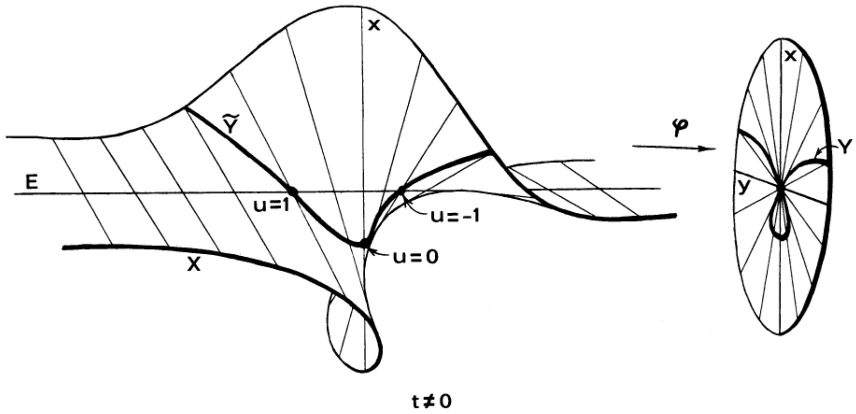

Figure 1: Blowing-up, figure taken from Hartshorne.

7. Define the curve $C := V(Y^2 - X^3 - X) \subset \mathbb{A}^2$.
   (a) Find a generator α such that $K(C) \cong K(X)(\alpha)$.
   (b) Consider the injection $K(X^2) \hookrightarrow K(C)$. Compute the corresponding dominant rational map.
8. Compute the Zariski cotangent space $\mathfrak{m}_{C,x}/\mathfrak{m}_{C,x}^2$ at $x = (0,0)$ for the following curves:
   (a) $C := V(Y^2 - X^3) \subset \mathbb{A}^2$.
   (b) $C := V(Y^2 - X^3 - X) \subset \mathbb{A}^2$.
9. Let A, B be integral domains. Show that a ring homomorphism $f : A \to B$ is injective if and only if the corresponding morphism $f^* : \mathrm{spec}(B) \to \mathrm{spec}(A)$ has dense image.