Algebraic Geometry

Exercise Sheet 3

Sheaves & Schemes

Note: If you have never seen the concept of direct limit, please have a look at the additional exercise A on the last page.

1. *Inverse Image Sheaf.* Let $f : X \to Y$ be a continuous map of topological spaces. For a sheaf $\mathscr{G}$ of abelian groups on Y we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{G}(V)$, where U is any open set of X and the direct limit (see exercise A) is taken over all open subsets V of Y containing $f(U)$. Prove that for every sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Prove that this induces a natural bijection of sets
$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$
for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. One says that f^{-1} and f_* are adjoint functors.

2. Let X be a topological space and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on X.
   (a) Prove that the induced map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open subset $U \subset X$ if and only if the map on the stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for every point $x \in X$.
   (b) Show that this is not true for surjectivity: Let X be the topological space $X := \mathbb{C} \setminus \{0\}$ with standard topology, let $\mathscr{F} = \mathscr{G}$ be the sheaf of nowhere-zero continuous complex-valued functions and let $\varphi : \mathscr{F} \to \mathscr{G}$ be the morphism that sends a function f to f^2. Prove that for every point $x \in X$ the induced morphism on the stalks φ_x is surjective, but on global sections $\varphi(X)$ is not surjective.

3. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Prove that if f is a homeomorphism and $f^\sharp$ is an isomorphism, then $(f, f^\sharp)$ is an isomorphism.

4. Let A be a ring and set $X := \mathrm{Spec}(A)$. Let $f \in A$ and let $U_f \subset X$ be the open complement of $V((f))$.
   (a) Show that the locally ringed space $(U_f, \mathcal{O}|_{U_f})$ is isomorphic to $\mathrm{Spec}(A_f)$.
   (b) For another element $g \in A$ describe the restriction map $\mathcal{O}(U_f) \to \mathcal{O}(U_{fg})$ in terms of a ring homomorphism $A_f \to A_{fg}$.

5. Consider $S_1 := \mathrm{Spec}(\mathbb{Q}[X,Y]/(XY))$ and $S_2 := \mathrm{Spec}(\mathbb{Q}[X,Y]/(X^2+Y^2))$.
   (a) Compute $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbb{Q}), S_i)$ for $i = 1, 2$.
   (b) Deduce that S_1 and S_2 are not isomorphic schemes.
   (c) Let $S_1' := \mathrm{Spec}(\mathbb{Q}(i)[X,Y]/(XY))$ and $S_2' := \mathrm{Spec}(\mathbb{Q}(i)[X,Y]/(X^2+Y^2))$. Prove that $S_1' \cong S_2'$ as schemes.

6. Let X be a scheme. For any point $x \in X$ we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k and let $k[\varepsilon]/(\varepsilon^2)$ be the *ring of dual numbers* over k. Show that to give a morphism of schemes over k of $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ to X is equivalent to giving a point $x \in X$ such that $k(x) = k$ and an element of T_x.

7. *The Proj-construction.* Let $S = \bigoplus_{d \geqslant 0} S_d$ be a graded ring. Denote $S_+ = \bigoplus_{d>0} S_d$. We define the set $\mathrm{Proj}(S)$ as the set of all homogeneous prime ideals $\mathfrak{p} \subset S$ which do not contain all of S_+. For a homogeneous ideal $\mathfrak{a} \subset S$ we define the subset $V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supset \mathfrak{a}\}$. These sets form the closed sets of a topology on $\mathrm{Proj}(S)$. We define a sheaf of rings $\mathcal{O}$ on $\mathrm{Proj}(S)$ as follows: For each $\mathfrak{p} \in \mathrm{Proj}(S)$ we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system of all homogeneous elments of S which are not in $\mathfrak{p}$. For any open subset $U \subset \mathrm{Proj}(S)$ we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$ we have $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a, f in S of the same degree such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Prove the following:
   (a) For any $\mathfrak{p} \in \mathrm{Proj}(S)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.
   (b) For any homogeneous $f \in S_+$ let U_f^+ be the complement of $V((f))$. These open sets cover $\mathrm{Proj}(S)$ and there is an isomorphism of locally ringed spaces
   $$(U_f^+, \mathcal{O}|_{U_f^+}) \cong \mathrm{Spec}(S_{(f)})$$
   where $S_{(f)}$ is the subring of elements of degree zero in the localized ring S_f.

   In particular $\mathrm{Proj}(S)$ is a scheme.

8. We define projective n-space over a ring A as $\mathbb{P}_A^n := \mathrm{Proj}(A[X_0, \ldots, X_n])$ with the definition of the previous exercise. Prove that $\mathbb{P}_A^n$ can also be seen as $n+1$ copies of affine n-space $\mathrm{Spec}(A[Y_1, \ldots, Y_n])$ glued together in the right way. Concretely, find an open cover of $n+1$ subsets of $\mathbb{P}_A^n$ which are all isomorphic to a copy of affine n-space $\mathrm{Spec}(A[Y_1, \ldots, Y_n])$. What are the transition morphisms between those charts?

A. *Direct Limit.* Let I be a partially ordered directed set, i.e. I is partially ordered and for every pair $i, j \in I$ there exists $k \in I$ such that $i \geqslant k$ and $j \geqslant k$. Let $(A_i)_{i \in I}$ be a family of abelian groups indexed by I. Suppose that for each pair $i, j \in I$ such that $i \geqslant j$ there exists a homomorphism $\mu_{ij} : A_i \to A_j$ satisfying the following two properties:

- For all $i \in I$ we have $\mu_{ii} = \mathrm{id}_{A_i}$.
- For $i, j, k \in I$ with $i \geqslant j \geqslant k$ we have $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$.

We call this collection (A_i, μ_{ij}) a direct system of abelian groups over I. Let C be the direct sum of all A_i and denote by D the $\mathbb{Z}$-submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ for $x_i \in A_i$ and $i, j \in I$ with $i \geqslant j$. We define the *direct limit* of the direct system (A_i, μ_{ij}), denoted by $\varinjlim A_i$ as the quotient $\mathbb{Z}$-module C/D. For every $i \in I$ there is a homomorphism $\nu_i : A_i \to \varinjlim A_i$. Prove the following properties of $\varinjlim A_i$:

(a) Elements $x \in \varinjlim A_i$ are equivalence classes containing (i, x_i) with $x_i \in A_i$ such that $(i, x_i) \sim (j, x_j)$ if and only if there exists a $k \in I$ with $i \geqslant k$ and $j \geqslant k$ and such that $\mu_{ik}(x_i) = \mu_{ik}(x_i)$.

(b) If each A_i is a ring and the homomorphisms are ring homomorphisms, then the direct limit has a natural ring structure such that ν_i is a ring homomorphism for all $i \in I$.

(c) An element $[(i, x_i)] \in \varinjlim A_i$ is zero if and only if there exists a $j \in I$ with $i \geqslant j$ such that $\mu_{ij}(x_i) = 0$.

(d) If B is a group and there are homomorphisms $\varphi_i : A_i \to B$ for all $i \in I$ such that $\varphi_i = \varphi_j \circ \mu_{ij}$ for all $i, j \in I$ with $i \geqslant j$, then there is a homomorphism $\varphi : \varinjlim A_i \to B$ such that $\varphi \circ \nu_i = \varphi_i$ for all $i \in I$.

In view of the above we conclude that the stalk of a sheaf $\mathscr{F}$ at a point x in a topological space X is the same as the direct limit over the rings $\mathscr{F}(U)$ for U an open neighbourhood of x.