D-MATH
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Algebraic Geometry

FS 2018

Exercise Sheet 4

Schemes

1. Consider the affine plane $\mathbb{A}^2_{\mathbb{R}}$ over $\mathbb{R}$. Show that the non-closed points are all either

   (a) the point (0), whose closure is all of $\mathbb{A}^2_{\mathbb{R}}$, or

   (b) the point (f) corresponding to an irreducible polynomial $f \in \mathbb{R}[X, Y]$.

   The polynomials of (b) may or may not remain irreducible in $\mathbb{C}[X, Y]$, so that a non-closed point in $\mathbb{A}^2_{\mathbb{R}}$ corresponds either to a single non-closed point of $\mathbb{A}^2_{\mathbb{C}}$ or to two non-closed points in $\mathbb{A}^2_{\mathbb{C}}$. The closed points in the closure of such a non-closed point may be either of both types above, or only of the second. Give examples of all these possibilities.

2. An inclusion of fields $K \hookrightarrow L$ induces a morphism $\mathbb{A}^n_L \to \mathbb{A}^n_K$. Find the images of the following points under the morphism $\mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$:

   (a) $(X - \sqrt{2}, Y - \sqrt{2})$

   (b) $(X - \sqrt{2}, Y - \sqrt{3})$

   (c) $(X - \zeta, Y - \zeta^{-1})$, for ζ a p-th root of unity for p prime

   (d) $(\sqrt{2}X - \sqrt{3}Y)$

   (e) $(\sqrt{2}X - \sqrt{3}Y - 1)$

3. *Double Points.* Let k be a field and $Y \hookrightarrow \mathbb{A}^2_k$ be a closed subscheme with image containing the origin $(0, 0)$ in $\mathbb{A}^2_k$ and such that $\mathcal{O}_Y(Y) \cong k[\varepsilon]/(\varepsilon^2)$. Denote by $\varphi : k[x, y] \to \mathcal{O}_Y(Y)$ the surjection defining the inclusion $Y \hookrightarrow \mathbb{A}^2$. Prove that the kernel of φ contains a non-zero element $\alpha x + \beta y$ for some $\alpha, \beta \in k$. Write $X_{\alpha,\beta} := \operatorname{Spec}(k[x, y]/\ker(\varphi))$ and show that $X_{\alpha,\beta}$ can also be characterized as the composition of the natural morphism $\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) \to \operatorname{Spec}(k[\varepsilon]) \cong \mathbb{A}^1_k$ with the inclusion of the line $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$ given by $x \mapsto (\beta x, -\alpha x)$.

4. Let k be an algebraically closed field and let $Z := \operatorname{Spec}(k[X_1, \ldots, X_n]/I) \subset \mathbb{A}^n_k$ be a closed subscheme of dimension 0 supported at the origin (i.e. $\sqrt{I} = (X_1, ..., X_n)$). Furthermore, suppose that $k[X_1, \ldots, X_n]/I$ is a 3-dimensional k-vector space. Prove that Z is isomorphic to either $A := \operatorname{Spec}(k[X]/(X^3))$ or to $B := \operatorname{Spec}(k[X, Y]/(X^2, XY, Y^2))$ and that A and B are not isomorphic to each other.

5. Let $X := \mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \subset \mathbb{A}^2_{\mathbb{C}}$. Prove:

   (a) The restriction map $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}(\mathbb{A}^2_{\mathbb{C}}) \to \mathcal{O}_X(X)$ is an isomorphism.

   (b) The scheme X is not an affine scheme.

6. Let X be a scheme and $f \in \mathcal{O}_X(X)$ a global section. Define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

   (a) If $U = \mathrm{Spec}(B)$ is an open affine subscheme of X and if $\bar{f} \in B = \mathcal{O}_U(U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

   (b) Assume that X is quasi-compact. Let $A := \mathcal{O}_X(X)$ and let $a \in A$ be an element whose restriction to X_f is 0. Show that there exists an integer $n > 0$ such that $f^n a = 0$.

   (c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \mathcal{O}_{X_f}(X_f)$. Show that there exists an integer $n > 0$ such that $f^n b$ is the restriction of an element of A.

   (d) With the hypothesis of (c) conclude that $\mathcal{O}_{X_f}(X_f) \cong A_f$.

7. *A Criterion for Affineness.*

   (a) Let $f : X \to Y$ be a morphism of schemes and suppose that Y can be covered by open subsets U_i such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

   (b) A scheme X is affine if and only if there is a finite set of elements $f_1, \dots, f_r \in A := \mathcal{O}_X(X)$ such that the open subsets X_{f_i} defined in exercise 6 are affine and $f_1, \dots, f_r$ generate the unit ideal.

8. Let k be a field. We want to study $\mathrm{Spec}\, A$ where $A = k(u) \otimes_k k(v)$ is the tensor product of two purely trascendental extension of k of trascendence degree 1 (i.e. $k(u) \cong k(X)$ the field of rational function at coefficient in k).

   (a) We have $k(u) = T^{-1}k[u]$ where T is the multiplicative set made up of the non-zero elements of $k[u]$. Deduce from this that A is the localization of $k[u, v]$ with respect to the multiplicative set T' made up of the non-zero elements of the form $P(u)Q(v) \in k[u, v]$.

   (b) Let $\mathfrak{m}$ be a maximal ideal of $k[u, v]$. Show that there exist a $P(u) \in \mathfrak{m} \setminus \{0\}$. Deduce from this that $T' \cap \mathfrak{m} \neq \varnothing$.

   (c) Show that the maximal ideals of A are of the form gA with $g \in k[u, v] \setminus (k[u] \cup k[v])$ irreducible in $k[u, v]$.

   (d) Show that $\mathrm{Spec}\, A$ is an infinite set and that $\dim A = 1$.

9. In the following if X is a scheme we denote by $\mathrm{sp}(X)$ the underlying topological space of X. Let S be a scheme and $\pi : X \to S$, $\rho : Y \to S$ be S-schemes. Let $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ be the fiber product of sets defined by π and ρ, endowed with the topology induced by the product topology on $\mathrm{sp}(X) \times \mathrm{sp}(Y)$. We are going to study some property concerning the relation between $\mathrm{sp}(X \times Y)$ and $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.

   (a) Show that we have a canonical map $f : \mathrm{sp}(X \times_S Y) \to \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.

   (b) Show that f is surjective.

   (c) Let us consider the example $X = Y = \operatorname{Spec} \mathbb{C}$ and $S = \operatorname{Spec} \mathbb{R}$. Show that $X \times_S Y \cong \operatorname{Spec}(\mathbb{C} \oplus \mathbb{C})$ and that f is not injective.

   (d) Show that in the case of the previous Exercise, with $X = \operatorname{Spec} k(u)$, $Y = \operatorname{Spec} k(v)$ and $S = \operatorname{Spec} k$, the map f has infinite fibers.

   (e) Let $S = \operatorname{Spec} k$ be the spectrum of an arbitrary field. By studying the example $X = Y = \mathbb{A}^1_k$, show that the image of an open subset under f is not an open subset.