D-MATH | Algebraic Geometry | FS 2018

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Exercise Sheet 5

Projective Space, Divisors and the Picard group

1. Let $n \geqslant 1$ be an integer. Let A be a commutative ring. Prove that there is a canonical isomorphism $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) \to A$. [Hint: view the projective space as obtained by gluing $\mathrm{Spec}(A[x_0/x_i, \ldots, x_n/x_i])$ for $0 \leqslant i \leqslant n$.]

2. (a) Construct a natural bijection $\mathbb{P}^n_{\mathbb{Z}}(\mathbb{Z}) \to \mathbb{P}^n_{\mathbb{Q}}(\mathbb{Q})$.

   (b) For a prime number p, compute the number of elements of $\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p)$.

3. Let S be a scheme. For any scheme $X \to S$ over S, the *diagonal* $\Delta_{X/S}$ is the unique morphism $X \to X \times_S X$ such that $p_1 \circ \Delta_{X/S} = p_2 \circ \Delta_{X/S}$ is the identity morphism of X.

   (a) Suppose that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(B)$ are affine. Prove that $\Delta_{X/S}$ is a closed immersion, and prove that the image of $\Delta_{X/S}$ is the closed subscheme of $\mathrm{Spec}(A \otimes_B A)$ defined by the ideal generated by the set of elements of the form $a \otimes 1 - 1 \otimes a$. Let D be the image of $\Delta_{X/S}$; for any scheme T over S, describe the set of T-valued points $D(T)$.

   (b) Let Y be any scheme and f_1, f_2 morphisms from X to Y. Suppose that $\Delta_{Y/S}$ is a closed immersion. Prove that the set of $x \in X$ such that $f_1(x) = f_2(x)$ is closed. [Hint: construct a closed immersion, using base change, for which this set is the image.]

   (c) Let $S = \mathrm{Spec}(\mathbb{C})$. Let X be the scheme over S defined by gluing $U_1 = \mathrm{Spec}(\mathbb{C}[X_1])$ with $U_2 = \mathrm{Spec}(\mathbb{C}[X_2])$ by identifying $U_{1,2} = \mathrm{Spec}(\mathbb{C}[X_1, X_1^{-1}])$ and $U_{2,1} = \mathrm{Spec}(\mathbb{C}[X_2, X_2^{-1}])$ with the isomorphism $X_1 \mapsto X_2$. Describe $X \times_S X$ and deduce that $\Delta_{X/S}$ is not a closed immersion.

4. Let K be a field and $n \geqslant 1$. Let X be projective n-space over K and $K(X)$ its function field.

   (a) For any prime Weil divisor D on X, defined by the vanishing of an irreducible homogeneous polynomial g, let $\mathcal{L}(D)$ be the sheaf defined by

   $$U \mapsto \{f \in K(X) \mid g^{-1}f \text{ is defined on } U\}.$$

   Show that $\mathcal{L}(D)$ is invertible, and show that the map $D \mapsto \mathcal{L}(D)$ induces a group homomorphism $\varphi\colon \mathrm{Cl}(X) \to \mathrm{Pic}(X)$.

(b) Let $\mathcal{L}$ be an invertible sheaf on X. Prove that there exists an injective morphism $\mathcal{L} \to \mathcal{K}$, where $\mathcal{K}$ is the constant sheaf $U \mapsto K(X)$.

(c) Deduce that the morphism φ is an isomorphism.

(d) Compute $\varphi^{-1}(\mathcal{O}(1))$.

5. A polynomial $f \in \mathbb{Z}[X_0, X_1, Y_0, Y_1]$ is said to be bi-homogeneous of bi-degree (d_1, d_2) if all monomials that appear with non-zero coefficients are of the form

$$X_0^a X_1^b Y_0^c Y_1^d$$

with $a + b = d_1$, $c + d = d_2$. Let $X = \mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$. Let $U_{i,j}$ be the open subscheme of X isomorphic to $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ where X_i and Y_j and invertible.

(a) Let $f \in \mathbb{Z}[X, Y, U, V]$ be a non-constant bi-homogeneous of bi-degree (d_1, d_2). Construct, by suitable gluing, a closed subscheme Y_f of X such that

$$Y \cap U_{i,j} = \operatorname{Spec}(\mathbb{Z}[X_0/X_i, X_1/X_i, Y_0/Y_j, Y_1/Y_j]/f_{i,j})$$

for $0 \leqslant i, j \leqslant 1$, where

$$f_{i,j} = f(X_0/X_i, X_1/X_i, Y_0/Y_j, Y_1/Y_j).$$

(b) Prove that the divisor class group of $\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.