Exercise Sheet 6

Curves

1. Consider the projective curve $C \subset \mathbb{P}^2$ over $\mathbb{Q}$ defined by the homogeneous equation $X^2 + Y^2 + Z^2 = 0$. Show that it is isomorphic to $\mathbb{P}^1$ over $\bar{\mathbb{Q}}$ but not over $\mathbb{Q}$.

2. Let $y^2 = x^3 + ax + b$ be an elliptic curve. Find an explicit equation for the point $(x_1, y_1) + (x_2, y_2)$ when $(x_1, y_1) \neq (x_2, y_2)$.

3. Let E_1, E_2 be two elliptic curves over an algebraically closed field $K = \bar{K}$. Let $\varphi : E_1 \to E_2$ be a morphism satisfying $\varphi(0) = 0$. Prove that φ is a group homomorphism on closed points.

4. *Sheaf of Differentials.* Let K be a field and let A be a K-algebra. Consider the free A-module F generated by all symbols da for $a \in A$. Let I be the A-submodule generated by all expressions of the form $d(a + a') - da - da'$ for $a, a' \in A$ and $d(aa') = ada' - a'da$ for $a, a' \in A$ and dc for $c \in K$. We define the *module of differential forms* of A over K to be $\Omega_{A/K} := F/I$. This module comes with a derivation $d : A \to \Omega_{A/K}$ sending $a \in A$ to da.

   Let X be a scheme over K. We define the *sheaf of differentials* $\Omega_{X/K}$ of X over K as follows: Let $U_i = \mathrm{Spec}(A_i)$ be an open affine cover of X. Then we define $\Omega_{X/K}|_{U_i}$ as the sheaf of modules associated to the A_i-module $\Omega_{A_i/K}$. These sheaves glue together to give a sheaf on X. The derivations $d_i : A_i \to \Omega_{A_i/K}$ also glue together to give a derivation $d : \mathcal{O}_X \to \Omega_{X/K}$.

   Now assume that X is a curve over K. Prove that $\Omega_{X/K}$ is an invertible sheaf.

5. Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ with $\deg(f) \leqslant g + 1$.

   [Recall the formula $\deg(f^*D) = \deg(f)\deg(D)$.]

6. Let $f : C_1 \to C_2$ be a finite morphism of curves. Prove that $g(C_1) \geqslant g(C_2)$.

7. Let K be a field with $\mathrm{char}(K) \neq 2$. Let $f \in K[X]$ be a polynomial of degree $d \geqslant 1$ with nonzero discriminant. Let C_0/K be the affine curve given by the equation

$$C_0 : y^2 = f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$$

   and let g be the unique integer satisfying $d - 3 < 2g \leqslant d - 1$.

(a) Let C be the closure of the image of C_0 via the map

$$[1, x, x^2, \ldots, x^{g-1}, y] : C_0 \to \mathbb{P}^{g+2}.$$

Prove that C is smooth and that $C \cap \{X_0 \neq 0\}$ is isomorphic to C_0. The curve C is called a *hyperelliptic curve.*

(b) Let $f^*(v) = v^{2g+2} f(1/v)$. Show that C consists of two affine pieces

$$C_0 : y^2 = f(x) \qquad \text{and} \qquad C_1 : w^2 = f^*(v)$$

glued together via the maps

$$\begin{array}{cc} C_0 \to C_1, & C_1 \to C_0 \\ (x, y) \mapsto (1/x, y/x^{g+1}), & (v, w) \mapsto (1/v, w/v^{g+1}) \end{array}$$