D-MATH
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Algebraic Geometry

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Solutions Sheet 1

Classical Varieties

Let K be an algebraically closed field. All algebraic sets below are defined over K, unless specified otherwise.

1. Describe the Zariski topology on $V(XY) \subset \mathbb{A}^2$.

   Solution: The algebraic set $V(XY)$ consists of the union of the two coordinate axis $Y = 0$ and $X = 0$. The proper closed subsets are given by the whole X-axis, the whole Y-axis and subsets consisting of finitely many points.

2. Assume that $\mathrm{char}(K) \neq 2, 3$. Show that the polynomial $Y^2 - X^3 - X \in K[X,Y]$ is irreducible. Describe the Zariski topology on $V(Y^2 - X^3 - X) \subset \mathbb{A}^2$.

   Solution: We consider the polynomial as an element in the ring $(K[X])[Y]$. In the ring $K[X]$ the element X is prime and divides $X^3 + X$, but its square does not. Using Eisenstein's criterion for the irreducibility of a polynomial, we deduce that $Y^2 - X^3 - X$ is irreducible. We conclude that $V(Y^2 - X^3 - X) \subset \mathbb{A}^2$ is an irreducible algebraic variety. Since $K[X,Y]$ has Krull dimension 2 and $(Y^2 - X^3 - X)$ is a non-zero prime ideal the coordinate ring $\mathcal{O}(V(Y^2 - X^3 - X)) = K[X,Y]/(Y^2 - X^3 - X)$ and thus also the variety has dimension 1. Hence the only proper closed subsets are given by finitely many points.

3. Let $C_1, C_2 \subset \mathbb{A}^2$ be two algebraic curves where C_2 is irreducible. Show that either $C_1 \cap C_2$ is a finite set or $C_2 \subset C_1$.

   Solution: Let $\mathfrak{a}, \mathfrak{p} \subset K[X,Y]$ be ideals such that $V(\mathfrak{a}) = C_1$ and $V(\mathfrak{p}) = C_2$. Since C_2 is irreducible $\mathfrak{p}$ is a prime ideal. We have $V(\mathfrak{a}) \cap V(\mathfrak{p}) = V(\mathfrak{a} + \mathfrak{p})$. If $\mathfrak{a} \subset \mathfrak{p}$ we conclude that $C_2 \subset C_1$. Otherwise, the irreducible subsets of $C_1 \cap C_2$ are given by $V(\mathfrak{m})$, where $\mathfrak{m}$ are prime ideals which contain $\mathfrak{a} + \mathfrak{p} \supsetneq \mathfrak{p}$. Because of the dimension of $K[X,Y]$ we conclude that all such $\mathfrak{m}$ are maximal ideals and thus $V(\mathfrak{m})$ are points. Every algebraic set can be covered by finitely many irreducible subsets, hence $C_1 \cap C_2$ is the union of finitely many points.

4. Let $A \subset \mathbb{A}^n$ and $B \subset \mathbb{A}^m$ be two algebraic sets. Prove that their product set $A \times B \subset \mathbb{A}^{n+m}$ is algebraic, too.

   Solution: Let $\mathfrak{a} \subset K[X_1, \ldots, X_n]$ and $\mathfrak{b} \subset K[Y_1, \ldots, Y_m]$ be ideals such that $A = V(\mathfrak{a})$ and $B = V(\mathfrak{b})$. Let $f_1, \ldots, f_r \in \mathfrak{a}$ and $g_1, \ldots, g_s \in \mathfrak{b}$ be respective generators of the ideals. It is straightforward to see that $A \times B = V(f_1, \ldots, f_r, g_1, \ldots, g_s)$, where we extended tacitly to the ring $K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$.

5. Consider the group $\mathrm{SL}_n(\mathbb{C})$. Note that it is given by the vanishing set of a polynomial and is thus also an algebraic set in $\mathbb{A}^{n^2}_{\mathbb{C}}$, i.e. an algebraic group.

   Let $H \subset \mathrm{SL}_n(\mathbb{C})$ be a subgroup. Show that the Zariski closure $\overline{H}$ is still a subgroup of $\mathrm{SL}_n(\mathbb{C})$.

   [Hint: observe that the multiplication and inversion maps are morphisms.]

   Solution: Let $i : \mathrm{SL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ be the inversion morphism. Since H is a subgroup we have $i : H \to H$. But i is a homeomorphism for the Zariski topology and thus maps $i : \overline{H} \to \overline{H}$. Hence $\overline{H}$ is closed under inversion. Now consider the multiplication morphism with a fixed element $x \in H$ given by $\mathrm{SL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$, $a \mapsto xa$. This is a homeomorphism, too and hence $x\overline{H} = \overline{H}$ for all $x \in H$. We conclude that for all $x \in \overline{H}$ we have $Hx \subset \overline{H}$. By the same argument as before we conclude that $\overline{H}x \subset \overline{H}$. Hence $\overline{H}$ is closed under multiplication. It follows that $\overline{H}$ is a subgroup of $\mathrm{SL}_n(\mathbb{C})$.

6. Determine the Zariski closure for the following subsets:

   (a) $\{(x, \sin(x)) \mid x \in \mathbb{C}\} \subset \mathbb{A}^2_{\mathbb{C}}$

   (b) $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{C}) \subset \mathbb{A}^4_{\mathbb{C}}$

   [Hint: Consider the subgroups $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{pmatrix}$.]

   (c) $\{(a^2 - b^2, 2ab, a^2 + b^2) \mid a, b \in \mathbb{Z}\} \subset \mathbb{A}^3_{\mathbb{C}}$.

   Solution:

   (a) Denote the subset by A. We claim that $\overline{A} = \mathbb{A}^2_{\mathbb{C}}$. To prove this we compute the dimension. Since A is not a finite set, we conclude that the dimension of $\overline{A}$ must be 1 or 2. Assume that it is 1. Then $\overline{A}$ is a curve. However, for all real values $-1 \leqslant y \leqslant 1$ the intersection with the irreducible curve given by $C_y := \{(x, y) \mid x \in \mathbb{C}\}$ has infinitely many points. We conclude using exercise 3 that $C_y \subset \overline{A}$ for all real $-1 \leqslant y \leqslant 1$, so $\overline{A}$ contains infinitely many different curves. But $\overline{A}$ is an algebraic set and thus must be coverable by finitely many irreducible curves which is a contradiction. We conclude that the dimension of $\overline{A}$ is 2 and since $\mathbb{A}^2_{\mathbb{C}}$ is irreducible we have proven the claim.
   Aliter: Consider the associated ideal $I(\overline{A}) \subset K[X, Y]$. A polynomial $f \in I(\overline{A})$ must in particular vanish on all points $(x, \sin(x))$. Hence for any $x \in \mathbb{R}$ the polynomial $f(X, \sin(x)) \in K[X]$ has infinitely many roots $X = 2\pi\mathbb{Z} + x$. Thus $f(X, \sin(x)) = 0$. Since this holds for all $x \in \mathbb{R}$ and $\sin(\mathbb{R}) = [-1, 1]$ we conclude that $f = 0$. Hence $I(\overline{A}) = (0)$ and $\overline{A} = \mathbb{A}^2_{\mathbb{C}}$.

   (b) Since $\mathbb{Z}$ is infinite and $\mathbb{A}^1_{\mathbb{C}}$ is irreducible, we conclude that the closure $\overline{\mathbb{Z}} = \mathbb{A}^1_{\mathbb{C}}$. We deduce that the closure of the subgroup $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ and similarly the closure of $\begin{pmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix}$. From exercise 5 we know that $\overline{\mathrm{SL}_2(\mathbb{Z})}$ is a group and from the preceding we know that it contains $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix}$. Notice that we have

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+ab & c+abc+a \\ b & 1+bc \end{pmatrix}$$

Hence all matrices in $\mathrm{SL}_2(\mathbb{C})$ of the form $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $z \neq 0$ are contained in $\overline{\mathrm{SL}_2(\mathbb{Z})}$. This is a Zariski dense subset of $\mathrm{SL}_2(\mathbb{C})$, because $\mathbb{A}^4_{\mathbb{C}} \smallsetminus V(Z)$ is Zariski dense in $\mathbb{A}^4_{\mathbb{C}}$. We conclude that $\overline{\mathrm{SL}_2(\mathbb{Z})} = \mathrm{SL}_2(\mathbb{C})$.

(c) Let A be the subset in question. These points are all zeroes of the polynomial $X^2+Y^2-Z^2$, they are Pythagorian triples. Hence $\overline{A} \subset V(X^2+Y^2-Z^2)$. The polynomial $X^2+Y^2-Z^2$ is irreducible in $K[X,Y,Z]$, which can be seen by the Eisenstein criterion with $Y-Z$ in the ring $(K[Y,Z])[X]$. Therefore $V(X^2+Y^2-Z^2)$ is irreducible of dimension 2. We only need to show that $\overline{A}$ has dimension 2. Containing infinitely many points, we know that it does not have dimension 0. Assume that it has dimension 1. For all $c \in \mathbb{Z}$, the intersection of A with the irreducible curve given by $C_c := \{(c^2-b^2, 2cb, c^2+b^2) \mid b \in \mathbb{C}\}$ has infinitely many points. By exercise 3 we conclude that $C_c \subset \overline{A}$ for all $c \in \mathbb{Z}$, hence $\overline{A}$ contains infinitely many different curves, which proves that it cannot have dimension 1. Thus $\overline{A}$ has dimension 2 and it follows that it is equal to $V(X^2+Y^2-Z^2)$.

7. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism. This shows that not every morphism whose underlying map of topological spaces is a homeomorphism needs to be an isomorphism.

   Solution: We see that the image of φ is contained in $V(y^2-x^3)$. It is bijective with inverse $\varphi^{-1} : (a,b) \mapsto \frac{b}{a}$ for $a \neq 0$ and $\varphi^{-1}(0,0) = 0$. Closed sets in $\mathbb{A}^1$ are finitely many points and they are mapped via φ to finitely many points (i.e. closed sets) in $\mathbb{A}^2$, so φ is a closed map. Hence φ and φ^{-1} are both continuous. So φ is bijective and bicontinuous. However, it is not an isomorphism. To see this let $f : \mathbb{A}^1 \to k$ be the canonical regular function $t \mapsto t$. Note that $f \circ \varphi^{-1} : (a,b) \mapsto \frac{b}{a}$ is not regular at $a = 0$, hence φ^{-1} is not a morphism.

8. Let $Y \subset \mathbb{A}^3$ be the set $Y := \{(t, t^2, t^3) \mid t \in K\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$ and prove that the coordinate ring $\mathcal{O}(Y)$ is isomorphic to a polynomial ring in one variable over K.

   Solution: The ideal $I(Y)$ is equal to $(v-u^2, w-u^3) \subset K[u,v,w]$. We have the equality $Y = V(v-u^2, w-u^3)$. Note that $K[u,v,w]/(v-u^2, w-u^3) \cong$

$K[u,u^2,u^3] \cong K[u]$. This proves that $\mathcal{O}(Y)$ is isomorphic to a polynomial ring in one variable over K and furthermore that $(v-u^2, w-u^3)$ is a prime ideal of coheight 1, so Y is an irreducible affine variety of dimension 1.

9. *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0,\dots,a_r)\times(b_0,\dots,b_s)$ to $(\dots,a_ib_j,\dots)$ in lexicographic order, where $N = rs+r+s$. Show that ψ is well-defined and injective. It is called the *Segre embedding.* Prove that the image of ψ is a projective algebraic set in $\mathbb{P}^N$.

   Solution: The map ψ is homogeneous and thus well-defined. Denote for all indices $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant s$ by x_{ij} the coordinates of a point in $\mathbb{P}^N$. The points in the image of ψ satisfy $x_{ij}x_{kl} = x_{kj}x_{il}$. Let Y be the projective algebraic set defined by these equations. Let $Q := [x_{ij}] \in Y$ be a point. Then at least one coordinate $x_{k\ell}$ is non-zero. Using that we are in projective space we have

   $$Q = [x_{k\ell}x_{ij}] = [x_{i\ell}x_{kj}] = \psi([x_{0\ell} : \cdots : x_{r\ell}], [x_{k0} : \cdots : x_{ks}]).$$

   Which proves that $Y = \operatorname{im}(\psi)$, hence the image of ψ is a projective algebraic set. The above also provides a left inverse to ψ which proves that ψ is injective.

10. Consider the surface Q (i.e. variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw$.

    (a) Show that Q is equal to the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

    (b) Show that Q contains two families of lines (i.e. linear varieties of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the property that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$ then $M_t \cap M_u = \varnothing$ and for all t, u we have $L_t \cap M_u =$ one point.

    (c) Show that Q contains other curves besides these lines and deduce that the Zariski topology on Q is not homeomorphic via the Segre embedding to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$.

    Solution:

    (a) The image of the Segre embedding is given by

    $$\operatorname{im}(\psi) = \left\{ [a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1] \;\middle|\; [a_0 : a_1], [b_0 : b_1] \in \mathbb{P}^1 \right\}$$

    We see that every point $[x : w : z : y]$ in the image of ψ satisfies the equation $xy - zw = 0$. Conversely suppose that a point $[x : w : z : y] \in \mathbb{P}^3$ satisfies $xy - zw = 0$. If $x \neq 0$, then the image of $([x : z], [x : w])$ under the Segre embedding is the point $[xx : xw : xz : wz] = [x : w : z : y]$. Similarly for $w \neq 0$ take $([w : y], [x : w])$, for $z \neq 0$ take $([x : z], [z : y])$ and for $y \neq 0$ take $([w : y], [z : y])$ to prove that every point $[x : w : z : y]$ with $xy - zw = 0$ is always in the image of the Segre embedding. Thus the image of the Segre embedding is equal to $V(xy - zw) = Q$.

(b) For any $t \in \mathbb{P}^1$ we define L_t to be $\psi(\mathbb{P}^1 \times \{t\})$ and for any $u \in \mathbb{P}^1$ we define M_u to be $\psi(\{u\} \times \mathbb{P}^1)$. We have shown in exercise 9 that ψ is injective. Thus $L_t \cap L_u = \varnothing$ and $M_t \cap M_u = \varnothing$ for $t \neq u$. Furthermore it follows that $L_t \cap M_u = \psi(u, t)$, which is one point.

(c) The surface Q contains the curve

$$C := V(xy - zw, w - z) = \psi\left(\left\{([x:z],[x:z]) \;\middle|\; [x:z] \in \mathbb{P}^1\right\}\right).$$

By bijectivity of ψ, we know that $C \cap L_t$ and $C \cap M_t$ are both one point for all $t \in \mathbb{P}^1$, hence we conclude that C is a different curve. The curve C is closed in $\mathbb{P}^3$, but the set $\{([x:z],[x:z]) \mid [x:z] \in \mathbb{P}^1\}$ is not closed in the product topology of $\mathbb{P}^1 \times \mathbb{P}^1$, because it is the diagonal of a non-Hausdorff space.

11. Let $n, d > 0$ be integers. We denote by $M_0, \ldots, M_N$ all monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N := \binom{n+d}{n} - 1$. We define the d-uple embedding as the map $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ sending the point $a = (a_0, \ldots, a_n)$ to the point $(M_0(a), \ldots, M_N(a))$. Show that the d-uple embedding of $\mathbb{P}^n$ is an isomorphism onto its image.

[Hint: Look at the inverse map.]

Solution: For $0 \leqslant i, j \leqslant n$ denote by M_{ij} the monomial $x_i^{d-1}x_j$ and denote for a point $[b_0 : \cdots : b_N]$ in the image of ρ_d by b_{ij} the coordinate corresponding to the monomial M_{ij}. Note that at least one b_{ii} is non-zero. On the chart $b_{ii} \neq 0$, define the map $\varphi_i : [b_0 : \cdots : b_N] \mapsto [b_{i0} : \cdots : b_{in}]$. These maps φ_i glue to an inverse of ρ_d on the whole image. Since they are defined only by projecting to certain coordinates, they are regular and thus glue to a morphism. Hence ρ_d is an isomorphism onto its image.