D-MATH | Algebraic Geometry | FS 2018
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Solutions Sheet 2

Classical Varietes, Rational Maps, Blowups, Spectrum

Let K be an algebraically closed field. All algebraic sets and varieties below are defined over K, unless specified otherwise.

1. Consider the set $M := \mathrm{Mat}_{m,n}(K)$ of $m \times n$-matrices. It can be identified with the affine algebraic variety $\mathbb{A}^{nm}$. Determine if S is open/closed/dense in M:

   (a) $S := \{A \in M \mid A^t A \text{ has an eigenvalue } 1\}$

   (b) $S := \{A \in M \mid \mathrm{rank}(A) = \min\{m, n\}\}$

   (c) for $m = n$, $S := \{A \in M \mid A \text{ is diagonalisable}\}$

   Solution:

   (a) We define the map $\varphi : M \to \mathbb{A}^1$ to be $A \mapsto \det(A^t A - \mathrm{id})$. For a matrix $A \in M$ the matrix $A^t A$ has an eigenvalue 1 if and only if $\varphi(A) = 0$. The map φ is a polynomial in the coefficients of A and hence a morphism of algebraic varieties. We conclude that $S = \varphi^{-1}(0)$ is a closed subset of M.

   (b) We define $d := \min\{m, n\}$ and $N := \binom{\max\{m,n\}}{d}$. We define $\varphi : M \to \mathbb{A}^N$ as the map taking an $m \times n$-matrix A to all of its $d \times d$-minors. Then A has full $\mathrm{rank}(A) = d$ if and only if $\varphi(A) \neq (0, \dots, 0)$. Since minors are polynomial expressions in the coefficients of A, we conclude that φ is a morphism of algebraic varieties. Hence $S = M \smallsetminus \varphi^{-1}(0)$ is an open subset of M.

   (c) It is neither: We define the map $\varphi : M \to \mathbb{A}^1$ as the map taking a matrix $A \in M$ to the discriminant of its characteristic polynomial. Since the discriminant is a polynomial expression in the coefficients of the characteristic polynomial, and these coefficients are polynomial expressions in the coefficients of A, we conclude that φ is a morphism of algebraic varieties. Furthermore, every matrix A with $\varphi(A) \neq 0$ is diagonalisable. Thus S contains an open set $S \supset \varphi^{-1}(0)$ and since M is irreducible, we conclude that S is dense in M. Since not all matrices are diagonalisable, we conclude that S is not closed in M. To see that S is neither open, for all $\alpha \in K$ define the matrices A_α as everywhere zero except for the upper left corner, where we have $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Define the set R to be the set of matrices A_α for $\alpha \neq 0$. Then every element of R is not diagonalisable and thus in the complement of S. However, the matrix A_0 is contained in the Zariski closure of R and is diagonalisable. Hence the complement of S is not closed, which implies that S is not open.

2. Construct a morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$ and a closed subvariety $Z \subset \mathbb{A}^2$ such that $f(Z)$ is not closed.

   Solution: Let f be the projection onto the first coordinate. We let $Z = V(XY-1)$. Then $f(Z) = \mathbb{A}^1 \setminus \{0\}$, which is not a closed subset of $\mathbb{A}^1$.

3. Recall the quadric surface Q given by $xy - zw$ in $\mathbb{P}^3$ of exercise 10, sheet 1. Prove that Q is birationally equivalent to $\mathbb{P}^2$.

   Solution: Let $U \subset Q$ be the open set defined as the complement $Q \setminus \{[0:0:0:1]\}$. We define the map $f : U \to \mathbb{P}^2$ as $[x:y:z:w] \mapsto [x:y:z]$. This is well-defined on U and thus defines a rational map from Q to $\mathbb{P}^2$. Let $V \subset \mathbb{P}^2$ be the open set where the third coordinate does not vanish. Define the map $g : V \to Q$ as $[x_0 : x_1 : x_2] \mapsto [x_0x_2 : x_1x_2 : x_2^2 : x_0x_1]$. It is well-defined on V and thus defines a rational map from $\mathbb{P}^2$ to Q. We note that the compositions $f \circ g$ and $g \circ f$ are the identity on the respective open sets where they are defined. Hence Q and $\mathbb{P}^2$ are birationally equivalent.

4. A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona transformation*. Define the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ as $[a_0 : a_1 : a_2] \mapsto [a_1a_2 : a_0a_2 : a_0a_1]$.

   (a) Show that φ is birational, and its own inverse.

   (b) Find open sets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

   (c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms.

   Solution:

   (a) Define the open set $V \subset \mathbb{P}^2$ to be the complement in $\mathbb{P}^2$ of the three points $[1:0:0], [0:1:0], [0:0:1]$. Then φ is regular on V, hence a rational map on $\mathbb{P}^2$. Let $[a_0 : a_1 : a_2]$ be a point in the preimage $\varphi^{-1}(V) \subset V$. Then

   $$\varphi^2([a_0 : a_1 : a_2]) = [a_0^2a_1a_2 : a_0a_1^2a_2 : a_0a_1a_2^2] = [a_0 : a_1 : a_2].$$

   So φ is birational and its own inverse.

   (b) Define the subset $U := \mathbb{P}^2 \setminus V(xyz)$ as the set of all points in $\mathbb{P}^2$ where no coordinate is zero. Then $\varphi(U) \subset U$ and by the above calculation, φ is an isomorphism from U to U. On U, the isomorphism φ is given by $[a_0 : a_1 : a_2] \mapsto [\frac{1}{a_0} : \frac{1}{a_1} : \frac{1}{a_2}]$.

   (c) The maps φ and $\varphi^{-1} = \varphi$ are both maximally defined on the open set V given in the solution of (a).

5. *Blowing-up.* We define the *Blowing-up* of $\mathbb{A}^2$ at the point 0 to be the subset $B := \{((x,y),[t:u]) \mid xu = ty\} \subset \mathbb{A}^2 \times \mathbb{P}^1$. Let $\varphi : B \to \mathbb{A}^2$ be the restriction to B of the projection onto the first component (see Figure 1). Prove that:

(a) The map φ is birational and restricts to an isomorphism $B \setminus \varphi^{-1}(0) \cong \mathbb{A}^2 \setminus 0$.

(b) We have $\varphi^{-1}(0) \cong \mathbb{P}^1$.

(c) The points in $\varphi^{-1}(0)$ are in 1-to-1-correspondence with lines ℓ in $\mathbb{A}^2$ through the point 0. [Hint: Look at $\varphi^{-1}(\ell \setminus 0)$ and its closure.]

Solution:

(a) The projection is a morphism and thus in particular a rational map. Define the open set $U := \mathbb{A}^2 \setminus 0$ and the morphism $\psi : U \to B$ given by $(x, y) \mapsto ((x, y), [x : y])$. The map ψ defines a rational map from $\mathbb{A}^2$ to B which is clearly the inverse of φ. Therefore φ is birational. Looking at the definition of ψ, we see that $\psi(U) \subset B \setminus \varphi^{-1}(0)$. Thus φ restricts to an isomorphism with inverse ψ.

(b) Since φ is projection onto the first component and $(0, [t : u]) \in B$ for all $[t : u] \in \mathbb{P}^1$, it follows that $\varphi^{-1}(0) = \{0\} \times \mathbb{P}^1 \cong \mathbb{P}^1$.

(c) Let $\ell \subset \mathbb{A}^2$ be a line through the point 0 given by $\ell = \{(az, bz) \mid z \in \mathbb{A}^1\}$ for two parameters $a, b \in K$ which are not both zero, i.e. $[a : b] \in \mathbb{P}^1$. The inverse image $\varphi^{-1}(\ell \setminus 0) = \psi(\ell \setminus 0)$ is given by $\{((az, bz), [az : bz]) \mid z \in \mathbb{A}^1 \setminus 0\}$. But in this set we have $[az : bz] = [a : b]$ which is also defined for $z = 0$. Hence the closure of $\varphi^{-1}(\ell \setminus 0)$ contains the point $((0, 0), [a : b])$. So we have a 1-to-1-correspondence given by sending ℓ to $((0, 0), [a : b])$.

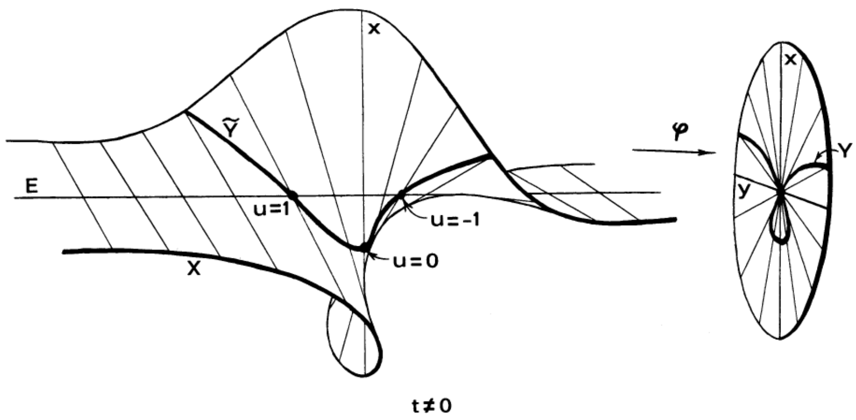

Figure 1: Blowing-up, figure taken from Hartshorne.

6. Use the same notation as in the previous exercise. Let C be the curve in $\mathbb{A}^2$ defined by the equation $y^2 = x^2(x + 1)$. Prove that C is singular at the point 0. We define the blowing-up of C to be the closure $\tilde{C} := \overline{\varphi^{-1}(C \setminus 0)}$, see Figure 1. Prove that $\tilde{C}$ is a nonsingular curve. We have thus removed the singularity of C by replacing it with a birationally equivalent curve $\tilde{C}$.

Solution: A generator for the ideal of C is given by $x^3+x^2-y^2$. The Jacobian matrix of the curve C at a point $(x,y) \in C$ is given by $(3x^2+2x, -2y)$, which does not have full rank for $(x,y) = (0,0)$. Hence C is singular at 0. The inverse image $D := \varphi^{-1}(C \smallsetminus 0)$ is given by $\{((x,y),[t:u]) \mid y^2 = x^2(x+1),\ xu = ty,\ (x,y) \neq (0,0)\}$. We look at the affine chart $t \neq 0$. On this chart, D consists of the points (x,y,u) in $\mathbb{A}^3$ such that $y^2 = x^2(x+1)$ and $xu = y$. Substituting we get $x^2u^2 = x^2(x+1)$ and hence for $x \neq 0$ we have $u^2 = x+1$. Hence the closure of D in the affine chart $\mathbb{A}^3$ is given by $V(xu-y, u^2-x-1)$. Its Jacobian matrix is $\begin{pmatrix} u & -1 & x \\ -1 & 0 & 2u \end{pmatrix}$, which has full rank everywhere on $V(xu-y, u^2-x-1)$. Thus at every point of $\tilde{C} \cap \{t \neq 0\}$ the curve is non-singular.

We now look at the chart $u \neq 0$, where D is given by all points (x,y,t) in $\mathbb{A}^3$ such that $y^2 = x^2(x+1)$ and $x = ty$. Substituting, we get the equation $1 = t^2(ty+1)$ for $y \neq 0$. Thus the closure of D in this affine chart is given by $V(x-ty, t^3y+t^2-1)$. Again, this curve is non-singular since the Jacobian matrix $\begin{pmatrix} 1 & -t & -y \\ 0 & t^3 & 3t^2y+2t \end{pmatrix}$ has full rank at every point of D.

Since non-singularity is a local property, we conclude that $\tilde{C}$ is a non-singular curve.

7. Define the curve $C := V(Y^2 - X^3 - X) \subset \mathbb{A}^2$.

   (a) Find a generator α such that $K(C) \cong K(X)(\alpha)$.

   (b) Consider the injection $K(X^2) \hookrightarrow K(C)$. Compute the corresponding dominant rational map.

   Solution:

   (a) We have seen that $K(C)$ can be identified with the field of fractions of the coordinate ring $\mathcal{O}(C) = K[X,Y]/(Y^2 - X^3 - X)$. Hence if α is an element with $\alpha^2 = X^3 + X$, we have $K(C) = K(X)(\alpha)$.

   (b) We let $Z := X^2$ and $i : K(Z) \to K(C)$ be the inclusion. It induces a ring homomorphism of coordinate rings $K[Z] \to K[X,Y]/(Y^2 - X^3 - X)$. The corresponding rational map is its associated morphism $C \to \mathbb{A}^1$ given by $(x,y) \mapsto x^2$, which is in particular a dominant rational map.

8. Compute the Zariski cotangent space $\mathfrak{m}_{C,x}/\mathfrak{m}_{C,x}^2$ at $x = (0,0)$ for the following curves:

   (a) $C := V(Y^2 - X^3) \subset \mathbb{A}^2$.

   (b) $C := V(Y^2 - X^3 - X) \subset \mathbb{A}^2$.

   Solution:

(a) We know that the maximal ideal of the local ring at $x = (0,0)$ is given by $\mathfrak{m} := (X,Y) \subset \mathcal{O}(C) = K[X,Y]/(Y^2 - X^3)$. In this ring, we have $(X,Y)^2 = (X^2, XY, Y^2) = (X^2, XY)$, because $Y^2 = X^3$. Thus $\mathfrak{m}/\mathfrak{m}^2$ is a vector space generated by X, Y. It has dimension two because C is singular at $x = (0,0)$ which can be seen with the Jacobian matrix.

(b) The maximal ideal is given by $\mathfrak{m} := (X,Y) \subset \mathcal{O}(C) = K[X,Y]/(Y^2 - X^3 - X)$. We have $(X,Y)^2 = (X^2, XY, Y^2) = (X, Y^2)$, because $Y^2 - (Y^2 - X^3 - X) - X \cdot X^2 = X$. We conclude that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space generated by Y.

9. Let A, B be integral domains. Show that a ring homomorphism $f : A \to B$ is injective if and only if the corresponding morphism $f^* : \mathrm{spec}(B) \to \mathrm{spec}(A)$ has dense image.

Solution: Suppose that f is injective. Let $\mathfrak{p} \in \mathrm{spec}(A)$ be the point corresponding to the prime ideal $(0) \in A$. Since f is injective, we know that $f^{-1}(0) = (0)$ and hence $\mathfrak{p}$ is in the image of f^*. Since $\mathfrak{p}$ is a dense point in $\mathrm{spec}(A)$ we conclude that the image is dense.

Conversely suppose that the image of f^* is dense. Let $I \subset A$ be the kernel of f. Since B is an integral domain I is a prime ideal. The morphism f^* then factors as $\mathrm{spec}(B) \to \mathrm{spec}(A/I) \to \mathrm{spec}(A)$. Since f^* has dense image, we conclude that $\mathrm{spec}(A/I) \to \mathrm{spec}(A)$ has dense image. However, the image is $V(I)$ and so a closed set. We conclude that $V(I) = V(0)$ and hence $I = (0)$.